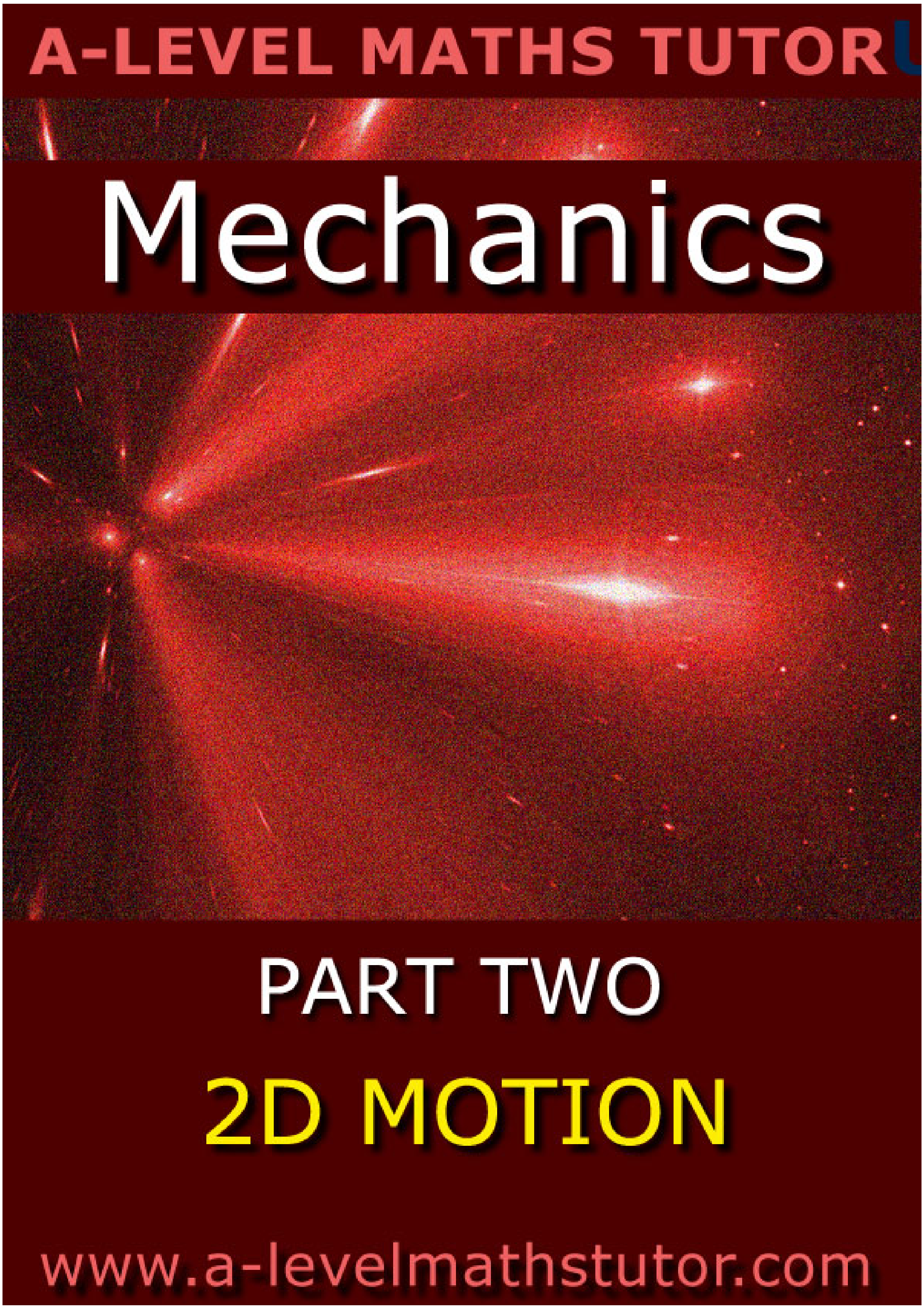
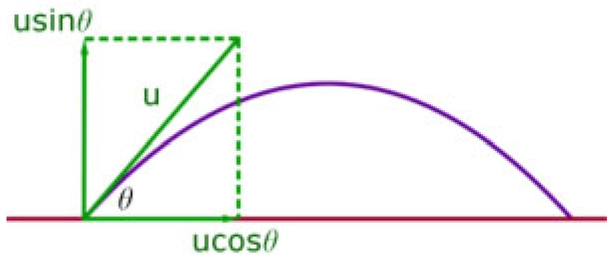

A-LEVEL MATHS TUTOR
Mechanics
PART TWO
2D MOTION
www.a-levelmathstutor.com

Contents

Projectiles

Vertical & horizontal components of velocity

When a particle is projected under gravity at a velocity u at an angle θ to the horizontal(neglecting air resistance)it follows the curve of a parabola.



The particle has an initial horizontal speed of ucosθ, which is unchanged throughout the motion.

Vertically the particle has an initial speed of usinθ. It falls under gravity and is accelerated downwards with an acceleration of g ms^{-1},where g = 9.8 ms^{-2} (approx.)

Time of flight

The time of flight is calculated from the vertical component of the velocity. It is the time it takes for the particle to go up, reach its maximum height and come down again. So this is twice the time to maximum height.

If the time to maximum height is t secs. Then the time of flight is 2t.

Consider motion up to maximum height. This is attained when the final velocity v = 0.

initial speed vertically upwards is $u\sin\theta$

using $v = u + at$

replacing u by $u\sin\theta$

substituting for acceleration $a = -g$

when $v = 0$

$$0 = u\sin\theta - gt$$

$$t = \frac{u\sin\theta}{g}$$

$\therefore$ time of flight($2t$) is $\dfrac{2u\sin\theta}{g}$

Maximum height attained (H)

The maximum height attained occurs when the particle is momentarily stationary, before falling under gravity. The vertical component of speed is zero at this point(v=0).

using $v^2 - u^2 = 2as$

final speed $v = 0$

u is replaced with $u\sin\theta$

distance s is height H

substituting for acceleration $a = -g$

$$0 - u^2\sin^2\theta = -2gH$$

$$-2gH = -u^2\sin^2\theta$$

$$H = \frac{u^2\sin^2\theta}{2g}$$

Range(R)

The range is simply the horizontal component of speed multiplied by the time of flight.

$$R = (u\cos\theta)t$$

Velocity(speed & direction) at time t

Solution of problems is to find the vertical component of speed at time t and combine this with the original horizontal component of speed, which remains unchanged.

Example

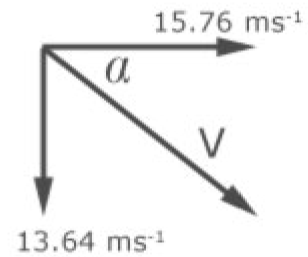
A particle P is projected at an angle of 45 degrees to the horizontal at a speed of 30 ms^{-1} . What is the speed and direction of the particle after 3 secs.?
(g=9.8 ms^{-2})

$$\text{constant horizontal speed} = 30\cos 45^\circ$$
$$= 15.760 \text{ ms}^{-1}$$
$$\text{initial vertical speed} \quad u = 30\cos 45^\circ$$
$$= 15.760 \text{ ms}^{-1}$$
v is vertical speed at $t = 3$ secs.

using $v = u + at$

substituting for $a = -g$

$$v = u - gt$$
$$= 15.76 - (9.8 \times 3)$$
$$= -13.64$$
vertical component of speed is -13.64 ms^{-1}



using Pythgoras, the speed V at time t is given by,

$$V^2 = (13.64)^2 + (15.76)^2$$
$$= 434.4272$$
$$\therefore \quad V = 20.8429$$
speed of particle after 3 secs. is 20.84 ms^{-1}

if the speed is inclined α deg. to the horiz.

$$\tan\alpha = \frac{13.64}{15.76} = 0.86548$$
$$\alpha = 40.8755^\circ$$
speed inclined at angle of 40.87° below horizontal

Circular Motion

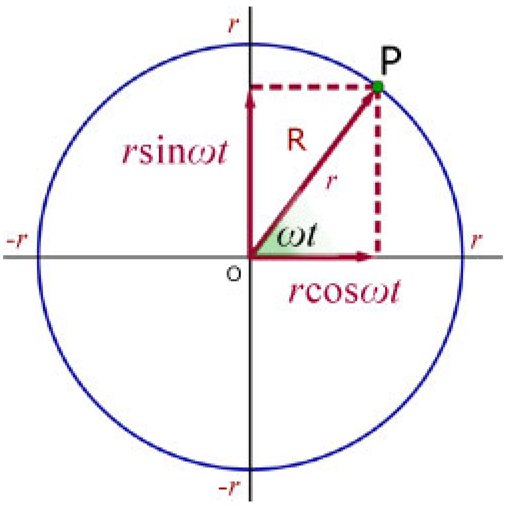
Summary of equations

$$\omega = \frac{\theta}{t} \qquad \omega = \frac{v}{r}$$

$$|\text{radial acceleration}| = \omega^2 r = \frac{v^2}{r}$$

$$|\text{tangential acceleration}| = \frac{dv}{dt}$$

Describing the circle - position vector **R**



i & **j** are unit vectors along the x and y-axis respectively.

The position vector **R** of a particle at P from O, at time t, is given by:

$$\mathbf{R} = (r\cos\omega t)\mathbf{i} + (r\sin\omega t)\mathbf{j}$$

As the position vector **R** rotates anti-clockwise, the particle at P traces out a circle of radius r .

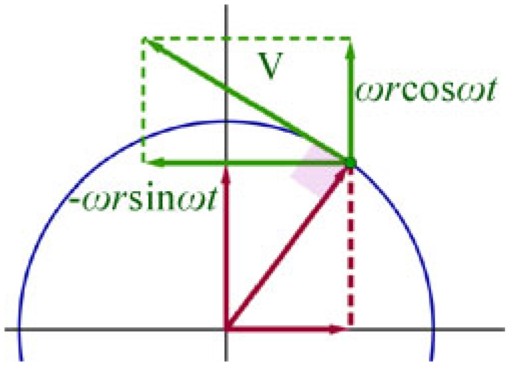
The velocity vector **V**

The velocity vector **V** at an instant is given by differentiating the position vector **R** with respect to *t* .

Here the unit vectors **i** & **j**, parallel to the x and y-axes, are centred on the particle at P.

$$\frac{d\mathbf{R}}{dt} = \mathbf{V}$$

$$\mathbf{V} = (-\omega r \sin \omega t)\mathbf{i} + (\omega r \cos \omega t)\mathbf{j}$$



the magnitude of the velocity is given by:

$$\begin{aligned}|\mathbf{V}| &= \sqrt{(-\omega r \sin \omega t)^2 + (\omega r \cos \omega t)^2} \\ &= \sqrt{\omega^2 r^2 \sin^2 \omega t + \omega^2 r^2 \cos^2 \omega t} \\ &= \sqrt{\omega^2 r^2 (\sin^2 \omega t + \cos^2 \omega t)}\end{aligned}$$

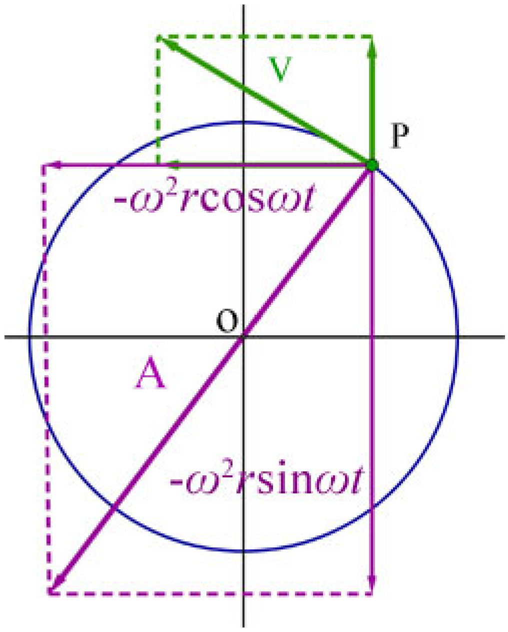
$$\text{but} \quad \sin^2 \omega t + \cos^2 \omega t = 1$$

$$\therefore \quad |\mathbf{V}| = \sqrt{\omega^2 r^2}$$

$$\underline{|\mathbf{V}| = \omega r}$$

The acceleration vector **A**

The acceleration of the particle at P is given by differentiating **V** with respect to t .



$$\mathbf{A} = \frac{d\mathbf{V}}{dt}$$

$$= (-\omega^2 r\cos\omega t)\mathbf{i} \ + \ (-\omega^2 r\sin\omega t)\mathbf{j}$$

but $\mathbf{R} = (r\cos\omega t)\mathbf{i} \ + \ (r\sin\omega t)\mathbf{j}$

$\therefore \quad \mathbf{A} = -\omega^2\mathbf{R}$

$\Rightarrow \quad \underline{|\mathbf{A}| = -\omega^2 r}$

Example

A satellite is moving at 2000 ms^{-1} in a circular orbit around a distant moon.
If the radius of the circle followed by the satellite is 1000 km, find:
i) the acceleration of the satellite
ii) the time for the satellite to complete one full orbit of the moon in minutes(2d.p.).

i) a is acceleration, v speed and r orbit radius,

$$a = \frac{v^2}{r}$$

substituting for $v = 2000\ \text{ms}^{-1}$, $r = 1000\ \text{km}(10^6\text{m})$

$$a = \frac{2000 \times 2000}{1000000} = 4$$

Ans. acceleration of the satellite is 4 ms^{-2}

ii) distance travelled by satellite in one orbit

$$= \text{circumference of orbit circle}$$

$$= 2\pi r = 2\pi \times 10^6$$

$$\text{time for one orbit} = \frac{\text{dist. travelled in one orbit}}{\text{speed}, v}$$

$$= \frac{2\pi \times 10^6}{2 \times 10^3} = 10^3 \pi$$

$$= 3142 \text{ secs.} \equiv 52.3\dot{6} \text{ mins.}$$

Ans. time for one orbit is $52.3\dot{6}$ mins.

Non-uniform circular motion(vertical circle)

A more in-depth treatment of motion in a vertical circle is to be found in 'kinetics/more circular motion'.

Here we look at the more general case of the acceleration component along the circle and the component towards the centre varying.

$$a_{\text{towards centre}} = \frac{v^2}{r}$$

$$a_{\text{along tangent}} = \frac{dv}{dt}$$

Example

A particle starts to move in a circular direction with an angular speed of 5 rad s^{-1} .
The radius of the circle of motion is 4 m, and the angular speed at time *t* is given by,

$$\omega = 15 - 3t$$

What is,

i) the linear speed of the particle 6 secs. after it starts moving?

ii) the resultant particle acceleration?

(answers to 1 d.p.)

i) $r = 4$ m $\quad t = 6$ secs.

$$\omega = \frac{v}{r}\ , \quad v = \omega r$$

substituting for $\omega = 15 - 3t$

$$v = r(15 - 3t)$$
$$= 4(15 - 3\times 6) = 4(15 - 18) = 4\times(-3)$$
$$= -12$$

Ans. linear speed of particle is 12.0 ms^{-1}(1d.p.)

ii) tangential acceleration component $= \dfrac{dv}{dt}$

$$\frac{dv}{dt} = \frac{d(\omega r)}{dt} = r\frac{d\omega}{dt}$$

substituting for $\omega = 15 - 3t$

$$\frac{dv}{dt} = r\frac{d(15-3t)}{dt} = -3r$$

substituting for $r = 4$ m

$$\frac{dv}{dt} = -3\times 4 = \underline{-12\ ms^{-2}}$$

radial acceleration component $= \dfrac{v^2}{r}$

$$\frac{v^2}{r} = \frac{12\times 12}{4} = \frac{144}{4} = \underline{36\ ms^{-2}}$$

resultant acceleration R (using Pythagoras),

$$R = \sqrt{(-12)^2 + (36)^2} = \sqrt{144 + 1296} = 37.9$$

Ans. resultant acceleration is magnitude 37.9 ms^{-2}

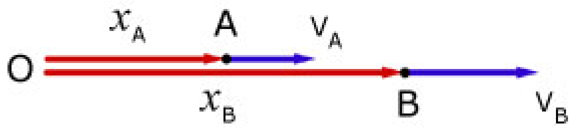
Relative Motion

One dimensional relative velocity(in a line)

Consider two particles A and B at instant t positioned along the x-axis from point O.

Particle A has a displacement $\boldsymbol{x}_A$ from O, and a velocity $\mathbf{V}_A$ along the x-axis. The displacement $\boldsymbol{x}_A$ is a function of time t .

Particle B has a displacement $\boldsymbol{x}_B$ from O, and a velocity $\mathbf{V}_B$ along the x-axis. The displacement $\boldsymbol{x}_B$ is a function of time t .



The velocity $\mathbf{V}_B$ relative to velocity $\mathbf{V}_A$ is written,

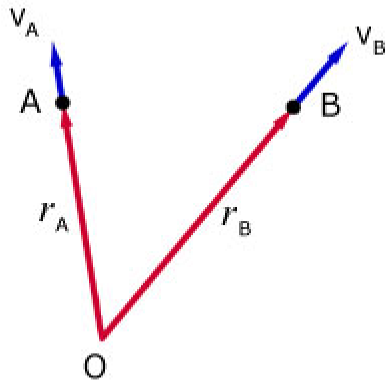
$${}_B\mathbf{V}_A = \mathbf{V}_B - \mathbf{V}_A$$

This can be expressed in terms of the derivative of the displacement with respect to time.

$$V_B = \frac{d(\boldsymbol{x}_B)}{dt} \qquad V_A = \frac{d(\boldsymbol{x}_A)}{dt}$$

$${}_BV_A = \frac{d(\boldsymbol{x}_B)}{dt} - \frac{d(\boldsymbol{x}_A)}{dt}$$

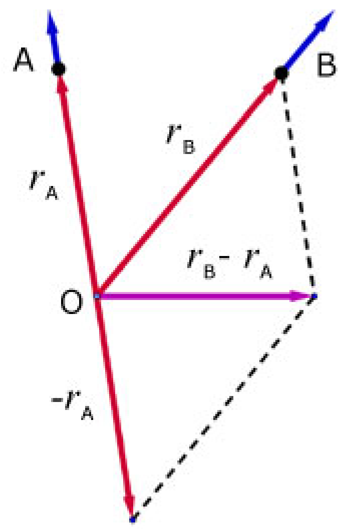
Two dimensional relative position & velocity



Particle A has a displacement $\boldsymbol{r}_A$ from O, and a velocity $\mathbf{V}_A$ along the x-axis. The displacement $\boldsymbol{r}_A$ is a function of time t .

Particle A has a displacement $\boldsymbol{r}_B$ from O, and a velocity $\mathbf{V}_B$ along the x-axis. The displacement $\boldsymbol{r}_B$ is a function of time t .

Relative position

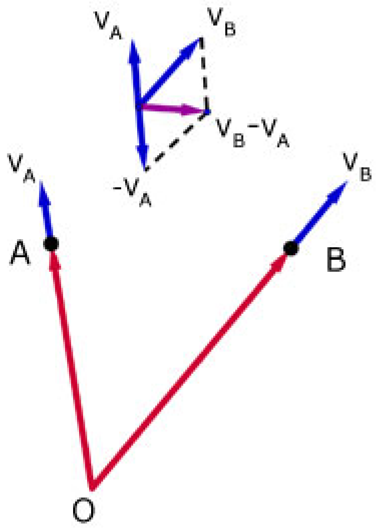


The position of B relative to A at time t is given by the position vector from O, $\boldsymbol{r}_{B\text{-}A}$.

The position vector $\boldsymbol{r}_{B\text{-}A}$ can be written as,

$${}_B\boldsymbol{r}_A = \boldsymbol{r}_B - \boldsymbol{r}_A$$

Relative velocity



Similarly, at time t the velocity vector $\mathbf{V}_B$ relative to velocity vector $\mathbf{V}_A$ can be written,

$${}_B\mathbf{V}_A = \mathbf{V}_B - \mathbf{V}_A$$

This can be expressed in terms of the derivative of the displacement with respect to time.

$$V_B = \frac{d(r_B)}{dt} \qquad V_A = \frac{d(r_A)}{dt}$$

$${}_BV_A = \frac{d(r_B)}{dt} - \frac{d(r_A)}{dt}$$

Example #1

If the velocity of a particle P is (9**i** - 2**j**) ms^{-1} and the velocity of another particle Q is (3**i** - 8**j**) ms^{-1} , what is the velocity of particle P relative to Q?

$$\begin{aligned} {}_PV_Q &= V_P - V_Q \\ &= (9\boldsymbol{i} - 2\boldsymbol{j}) - (3\boldsymbol{i} - 8\boldsymbol{j}) \\ &= 9\boldsymbol{i} - 3\boldsymbol{i} + 8\boldsymbol{j} - 2\boldsymbol{j} \\ &= (9\boldsymbol{i} - 3\boldsymbol{i}) + (8\boldsymbol{j} - 2\boldsymbol{j}) \\ &= 6\boldsymbol{i} + 6\boldsymbol{j} \end{aligned}$$

Ans. velocity of P relative to Q is $(6\boldsymbol{i} + 6\boldsymbol{j})$ ms^{-1}

Example #2

A particle P has a velocity (4**i** + 3**j**) ms^{-1}. If a second particle Q has a relative velocity to P of (2**i** - 3**j**), what is the velocity of Q?

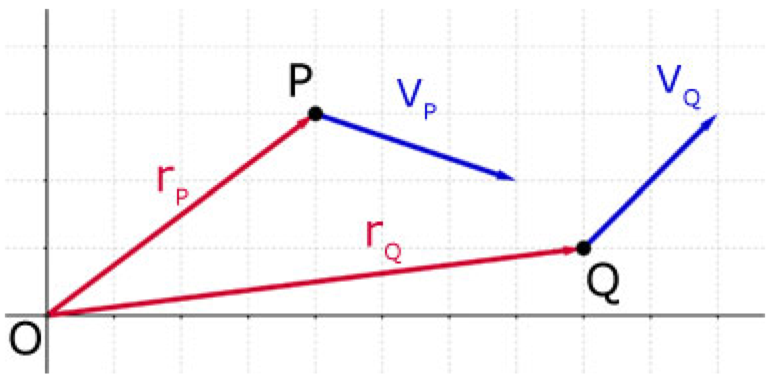
$$
\begin{aligned}
{}_Q V_P &= V_Q - V_P \\
V_Q &= {}_Q V_P + V_P \\
&= (2\boldsymbol{i} - 3\boldsymbol{j}) + (4\boldsymbol{i} + 3\boldsymbol{j}) \\
&= 2\boldsymbol{i} + 4\boldsymbol{i} - 3\boldsymbol{j} + 3\boldsymbol{j} \\
&= 6\boldsymbol{i}
\end{aligned}
$$

Ans. velocity of Q is $(6\boldsymbol{i})\ ms^{-1}$

Example #3

A radar station at O tracks two ships P & Q at 0900hours (t=0) .
P has position vector (4**i** + 3**j**) km, with velocity vector (3**i** - **j**) km hr^{-1}.
Q has position vector (8**i** + **j**) km, with velocity vector (2**i** + 2**j**) km hr^{-1}.

i) What is the displacement of P relative to Q at 0900 hours? (ie distance between ships). Answer to 2 d.p.
ii) Write an expression for the displacement of P relative to Q in terms of time t .
iii) Hence calculate the displacement of P relative to Q at 1500 hours.



i)

$$\boldsymbol{r}_\text{P} = (4\boldsymbol{i} + 3\boldsymbol{j}) \qquad \boldsymbol{r}_\text{Q} = (8\boldsymbol{i} + \boldsymbol{j})$$

$$\begin{aligned}\boldsymbol{r}_\text{P} - \boldsymbol{r}_\text{Q} &= (4\boldsymbol{i} + 3\boldsymbol{j}) - (8\boldsymbol{i} + \boldsymbol{j}) \\ &= 4\boldsymbol{i} + 3\boldsymbol{j} - 8\boldsymbol{i} - \boldsymbol{j} \\ &= 4\boldsymbol{i} - 8\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{j} \\ &= (-4\boldsymbol{i} + 2\boldsymbol{j})\end{aligned}$$

magnitude of the displacement, using Pythagoras,

$$\begin{aligned}\left|\boldsymbol{r}_\text{P} - \boldsymbol{r}_\text{Q}\right| &= \sqrt{(-4)^2 + (2)^2} \\ &= \sqrt{16 + 4} = \sqrt{20} = 4.47\end{aligned}$$

Ans. distance between ships at 0900 is 4.47 km.

ii)

$\boldsymbol{r}_{\text{P},0}$ displacement vector of P at time 0

$\boldsymbol{r}_{\text{P,t}}$ displacement vector of P at time t

$\boldsymbol{r}_{\text{Q},0}$ displacement vector of Q at time 0

$\boldsymbol{r}_{\text{Q,t}}$ displacement vector of Q at time t

$\left|\text{V}_\text{P}t\right|$ distance travelled by P in time t

$\left|\text{V}_\text{Q}t\right|$ distance travelled by Q in time t

$$\begin{aligned}\boldsymbol{r}_{\text{P,t}} &= \boldsymbol{r}_{\text{P},0} + \text{V}_\text{P}t \\ &= (4\boldsymbol{i} + 3\boldsymbol{j}) + (3\boldsymbol{i} - \boldsymbol{j})\end{aligned}$$

$$\begin{aligned}\boldsymbol{r}_{\text{Q,t}} &= \boldsymbol{r}_{\text{Q},0} + \text{V}_\text{Q}t \\ &= (8\boldsymbol{i} + \boldsymbol{j}) + (2\boldsymbol{i} - 2\boldsymbol{j})\end{aligned}$$

therefore the displacement of P relative to Q is given by,

$$
\begin{aligned}
{}_{P,t}\boldsymbol{r}_{Q,t} &= \boldsymbol{r}_{P,t} - \boldsymbol{r}_{Q,t} \\
&= (4\boldsymbol{i} + 3\boldsymbol{j}) + (3\boldsymbol{i} - \boldsymbol{j})t - [(8\boldsymbol{i} + \boldsymbol{j}) + (2\boldsymbol{i} + 2\boldsymbol{j})t] \\
&= 4\boldsymbol{i} + 3\boldsymbol{j} + 3\boldsymbol{i}t - \boldsymbol{j}t - 8\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{i}t - 2\boldsymbol{j}t \\
&= (4\boldsymbol{i} - 8\boldsymbol{i}) + (3\boldsymbol{j} - \boldsymbol{j}) + 3\boldsymbol{i}t - \boldsymbol{j}t - 2\boldsymbol{i}t - 2\boldsymbol{j}t \\
&= (-4\boldsymbol{i} + 2\boldsymbol{j}) + (3\boldsymbol{i} - 2\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{j})t \\
&= (-4\boldsymbol{i} + 2\boldsymbol{j}) + (\boldsymbol{i} - 3\boldsymbol{j})t
\end{aligned}
$$

$$\underline{{}_{P,t}\boldsymbol{r}_{Q,t} = (-4\boldsymbol{i} + 2\boldsymbol{j}) + (\boldsymbol{i} - 3\boldsymbol{j})t}$$

iii) using the result above for 1500 hours(t = 6)

$$
\begin{aligned}
{}_{P,t}\boldsymbol{r}_{Q,t} &= (-4\boldsymbol{i} + 2\boldsymbol{j}) + (\boldsymbol{i} - 3\boldsymbol{j})6 \\
&= -4\boldsymbol{i} + 2\boldsymbol{j} + 6\boldsymbol{i} - 18\boldsymbol{j} \\
&= -4\boldsymbol{i} + 6\boldsymbol{i} + 2\boldsymbol{j} - 18\boldsymbol{j} \\
&= 2\boldsymbol{i} - 16\boldsymbol{j} \\
\left|{}_{P,t}\boldsymbol{r}_{Q,t}\right| &= \sqrt{(2)^2 + (16)^2} \\
&= \sqrt{4 + 256} = \sqrt{260} = 16.12
\end{aligned}
$$

$$\underline{\text{Ans. displacement P relative Q at 1500 hours is 16.12 km}}$$

Two dimensional relative acceleration

Similarly, if $\mathbf{a}_A$ and $\mathbf{a}_B$ are the acceleration vectors at A and B at time t, then the acceleration of B relative to A is given by,

$$\mathbf{a}_B = \frac{d(\mathbf{V}_B)}{dt} \qquad \mathbf{a}_A = \frac{d(\mathbf{V}_A)}{dt}$$

$$_B\mathbf{a}_A = \frac{d(\mathbf{V}_B)}{dt} - \frac{d(\mathbf{V}_A)}{dt}$$

$$= \frac{d^2(\mathbf{r}_B)}{dt^2} - \frac{d^2(\mathbf{r}_A)}{dt^2}$$

Notes