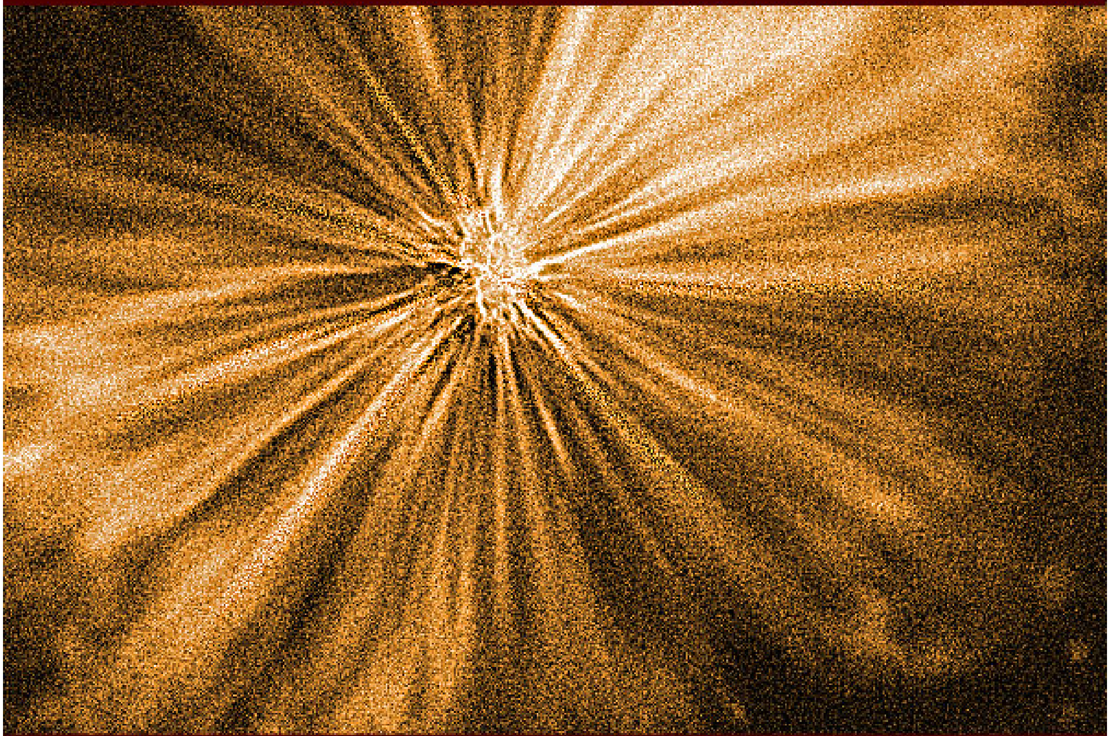
A-LEVEL MATHS TUTOR

Mechanics



PART ONE

LINEAR MOTION

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Contents

Uniform Acceleration

Introduction

To understand this section you must remember the letters representing the variables:

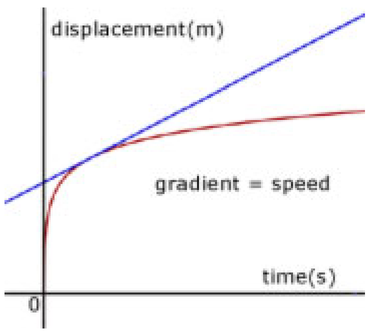
u - initial speed
v - final speed
a - acceleration(+) or deceleration(-)
t - time taken for the change
s - displacement(distance moved)

It is also important to know the **S.I. units** (*Le Système International d'Unités*) for these quantities:

u - metres per second (ms^{-1})
v - metres per second (ms^{-1})
a - metres per second per second (ms^{-2})
t - seconds (s)
s - metres (m)

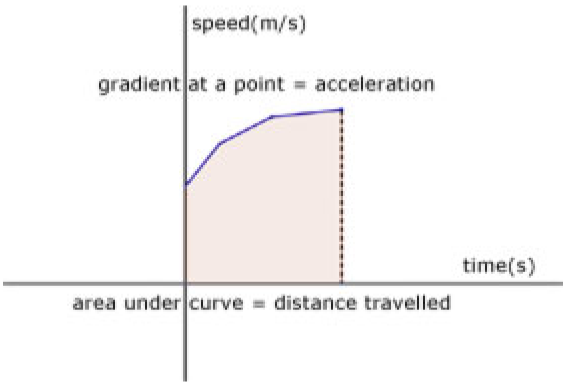
in some text books 'speed' is replaced with 'velocity'. Velocity is more appropriate when direction is important.

Displacement-time graphs



For a displacement-time graph, the gradient at a point is equal to the **speed** .

Speed-time graphs



For a speed-time graph, the area under the curve is the distance travelled.

The gradient at any point on the curve equals the acceleration.

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

Note, the acceleration is also the second derivative of a speed-time function.

Equations of Motion

One of the equations of motion stems from the definition of acceleration:

acceleration = the rate of change of speed

$$a = \frac{v - u}{t}$$

rearranging

$$v = u + at \qquad (1$$

if we define the distance 's' as the average speed times the time(t), then:

$$s = \left(\frac{u + v}{2}\right)t$$

rearranging

$$u + v = \frac{2s}{t}$$

rearranging (i

$$v - u = at$$

subtracting these two equations to eliminate *v*

$$2u = \frac{2s}{t} - at$$
$$2ut = 2s - at^2$$
$$2ut + at^2 = 2s$$
$$ut + \frac{at^2}{2} = s, \qquad s = ut + \frac{at^2}{2}$$

it is left to the reader to show that :

$$v^2 - u^2 = 2as$$

hint: try multiplying the two equations instead of subtracting summary:

$$s = \left(\frac{u+v}{2}\right)t$$
$$v - u = at$$
$$s = ut + \frac{at^2}{2}$$
$$v^2 - u^2 = 2as$$

Example #1

A car starts from rest and accelerates at 10 ms^{-1} for 3 secs.
What is the maximum speed it attains?

$u = 0 \text{ ms}^{-1} \qquad a = 10 \text{ ms}^{-2} \qquad t = 3 \text{ s}$

$$\begin{aligned} v &= u + at \\ &= 0 + (10 \times 3) \\ &= 30 \end{aligned}$$

Ans. maximum speed is 30 ms^{-1}

Example #2

A car travelling at 25 ms^{-1} starts to decelerate at 5 ms^{-2}.
How long will it take for the car to come to rest?

$u = 25 \text{ ms}^{-1} \qquad v = 0 \qquad a = -5 \text{ ms}^{-2}$

$$\begin{aligned} v &= u + at \\ 0 &= 25 + (-5)t \\ 5t &= 25 \\ t &= 5 \end{aligned}$$

Ans. it takes 5 secs. for the car to stop

Example #3

A car travelling at 20 ms^{-1} decelerates at 5 ms^{-2}.
How far will the car travel before stopping?

$$u = 20 \text{ ms}^{-1} \qquad v = 0 \qquad a = -5 \text{ ms}^{-2}$$

$$\begin{aligned} v^2 - u^2 &= 2as \\ 0 - (20 \times 20) &= 2(-5)s \\ -400 &= -10s \\ s = \frac{400}{10} &= 40 \end{aligned}$$

Ans. distance travelled is 40 m

Example #4

A car travelling at 30 ms^{-1} accelerates at 5 ms^{-2} for 8 secs.
How far did the car travel during the period of acceleration?

$$u = 30 \text{ ms}^{-1} \qquad a = 5 \text{ ms}^{-2} \qquad t = 8 \text{ s}$$

$$s = ut + \tfrac{1}{2}at^2$$

$$\begin{aligned} s &= (30 \times 8) + \tfrac{1}{2}(5 \times 8 \times 8) \\ &= 240 + 160 \\ &= 400 \end{aligned}$$

Ans. car travelled 400 m during acceleration

Vertical motion under gravity

These problems concern a particle projected vertically upwards and falling 'under gravity'.

In these types of problem it is assumed that:

air resistance is minimal

displacement & velocity are positive(+) upwards & negative(-)downwards

acceleration(g) always acts downwards and is therefore negative(-)

acceleration due to gravity(g) is a constant

Example #1

A stone is thrown vertically upwards at 15 ms^{-1}.

(i) what is the maximum height attained?
(ii) how long is the stone in the air before hitting the ground?

i)

(Assume g = 9.8 ms^{-2}. Both answers to 2 d.p.)

$$u = 15\ \text{ms}^{-1} \qquad a = -9.8\ \text{ms}^{-2}$$

max. height is when final velocity $v = 0$

$$v^2 - u^2 = 2as$$
$$0 - (15)^2 = 2(-9.8)s$$
$$-225 = -19.6s$$
$$s = \frac{225}{19.6} = 11.4796$$

Ans. max. height is 11.48 m (2 d.p.)

ii)

$$u = 15 \text{ ms}^{-1} \qquad a = -9.8 \text{ ms}^{-2}$$

time of flight is $2t$, twice time to max. height t

max. height is when final velocity $v = 0$

$$v = u + at$$
$$= 15 + (-9.8)t$$
$$15 = 9.8t$$
$$t = \frac{15}{9.8} = 1.5306$$
$$\therefore \quad 2t = 3.0612$$

Ans. max. time in air is 3.06 secs. (2 d.p.)

Example #2

A boy throws a stone vertically down a well at 12 ms^{-1}.
If he hears the stone hit the water 3 secs. later,

(i) how deep is the well?
(ii)what is the speed of the stone when it hits the water?

i)

(Assume g = 9.8 ms^{-2}. Both answers to 1 d.p.)

$$u = -12 \text{ ms}^{-1} \qquad a = -9.8 \text{ ms}^{-2} \quad t = 3 \text{ s}$$

$$s = ut + \tfrac{1}{2}at^2$$
$$= (-12)(3) + \tfrac{1}{2}(-9.8)(3 \times 3)$$
$$= -36 - 44.1$$
$$= -80.1$$

Ans. depth of well is 80.1 m (1 d.p.)

ii)

$$\begin{aligned} v &= u + at \\ &= (-12) + (-9.8)(3) \\ &= -121 - 29.4 \\ &= -41.4 \end{aligned}$$

<u>Ans. stone strikes water at $41.4\ \text{ms}^{-1}$ (1 d.p.)</u>

Non-uniform Acceleration

Theory

Consider a particle P moving in a straight line from a starting point O.

The displacement from O is x at time t .
The initial conditions are: $t \geq 0$ when x=0.
if v is the velocity of P at time t, then

$$v = \frac{dx}{dt}$$

The acceleration 'a' of particle P is defined as:

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

or alternately,

$$\begin{aligned} a &= \frac{dv}{dt} \\ &= \frac{dv}{dx}\frac{dx}{dt} \end{aligned}$$

$$\text{but} \quad \frac{dx}{dt} = v$$

$$\therefore \qquad a = v\frac{dv}{dx}$$

Problems on this topic are solved by analysing the information given to form a differential equation. This is then integrated, usually between limits.

Example #1

A particle moves in a straight line such that its acceleration 'a' at time 't' is given by:

$$a = 4t - 7$$

If the initial speed of the particle is 5 ms^{-1}, at what values of 't' is the particle stationary?

$$a = \frac{dv}{dt}$$

$$\Rightarrow \quad \frac{dv}{dt} = 4t - 7$$

$$\Rightarrow \quad dv = (4t - 7)dt$$

integrating both sides

$$\int dv = \int (4t - 7)dt$$

$$\Rightarrow \quad v = 2t^2 - 7t + C$$

but $v = 5 \text{ ms}^{-1}$ when $t = 0$

$$\Rightarrow \quad 5 = 0 - 0 + C$$

$$\therefore \quad C = 5$$

$$\Rightarrow \quad v = 2t^2 - 7t + 5$$

$$= (2t - 5)(t - 1)$$

the particle is at rest when $v = 0$

$$\therefore \quad (2t - 5)(t - 1) = 0$$

$$\Rightarrow \quad t = 5/2 \,, \qquad t = 1$$

particle is stationary at $t = 1$ sec. $t = 2.5$ sec.

Example #2

A particle moves from a point O in a straight line with initial velocity 4 ms^{-1}.
if v is the velocity at any instant, the acceleration a of the particle is given by:

$$a = \frac{3}{v}$$

The particle passes through a point X with velocity 8 ms^{-1}.

(i) how long does the particle take to reach point X?
(ii) what is the distance OX?(1 d.p.)

i)

$$a = \frac{dv}{dt}$$

$$\Rightarrow \qquad \frac{dv}{dt} = \frac{3}{v}$$

$$\Rightarrow \qquad dt = \frac{v}{3}dv$$

integrating both sides

$$\int dt = \int \frac{v}{3}dv$$

the limits of v are 8 ms^{-1} and 4 ms^{-1}

$$t = \left[\frac{v^2}{6}\right]_4^8$$

$$= \left[\frac{64}{6}\right] - \left[\frac{16}{6}\right]$$

$$= \frac{48}{6} = 8$$

Ans. particle takes 8 secs. to reach X

ii)

$$v\frac{dv}{dx} = \frac{3}{v}$$

$$\Rightarrow \qquad dx = \frac{v^2}{3}dv$$

integrating both sides

$$\int dx = \int \frac{v^2}{3}dv$$

the limits of v are 8ms^{-1} and 4ms^{-1}

$$x = \left[\frac{v^3}{9}\right]_4^8$$

$$= \left[\frac{512}{9}\right] - \left[\frac{64}{9}\right]$$

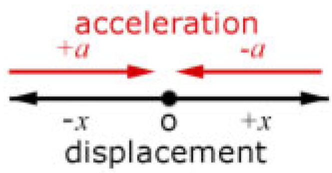
$$= \frac{448}{9} = 49.\dot{7}$$

Ans. distance OX is 49.8 metres (1 d.p.)

Simple Harmonic Motion

Theory

A particle is said to move with S.H.M when the acceleration of the particle about a fixed point is proportional to its displacement but opposite in direction.



Hence, when the displacement is positive the acceleration is negative(and vice versa).

This can be described by the equation:

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

where x is the displacement about a fixed point O(positive to the right, negative to the left), and w^2 is a positive constant.

An equation for velocity is obtained using the expression for acceleration in terms of velocity and rate of change of velocity with respect to displacement(see 'non-uniform acceleration').

$$v\frac{dv}{dx} = -\omega^2 x$$

separating the variable and integrating,

$$\int v\,dv = \int -\omega^2 dx$$
$$\tfrac{1}{2}v^2 = -\tfrac{1}{2}\omega^2 x^2 + C$$

$v = 0$ when $x = a$

$\Rightarrow \quad \underline{v^2 = \omega^2(a^2 - x^2)}$

$\Rightarrow \quad v = \pm\omega(a^2 - x^2)^{\frac{1}{2}}$

but $\quad v = \dfrac{dx}{dt}$

$\Rightarrow \quad \dfrac{dx}{dt} = \pm\omega(a^2 - x^2)^{\frac{1}{2}}$

separating the variable and integrating again,

$$\int \frac{dx}{(a^2 - x^2)^{\frac{1}{2}}} = \int \pm\omega\,dt$$
$$-\cos^{-1}\left(\frac{x}{a}\right) = \pm\omega t + C$$

when $t = 0$ when $x = a$

$\Rightarrow \quad C = 0$

$\therefore \quad \cos(\omega t) = \dfrac{x}{a} \quad$ or $\quad \underline{x = a\cos(\omega t)}$

NB $\cos^{-1}()$ is the same as arc cos()

So the displacement against time is a cosine curve. This means that at the end of one completete cycle,

$\omega T = 2\pi \qquad$ (T is the period)

$\therefore \quad \underline{T = \dfrac{2\pi}{\omega}}$

<u>Example</u>

A particle displaying SHM moves in a straight line between extreme positions A & B and passes through a mid-position O.

If the distance AB=10 m and the max. speed of the particle is 15 m^{-1} find the period of the motion to 1 decimal place.

$\mathrm{AB} = 10\mathrm{m} \quad \therefore \quad \text{amplitude} \ a = 5\mathrm{m}$

$\text{max. speed} \quad v_{max} = 15\mathrm{ms}^{-1}$

$$v^2 = \omega^2(a^2 - x^2)$$

when v is max., displacement $x = 0$

$\therefore \quad v^2 = \omega^2 a^2$

$\Rightarrow \quad v = \omega a$

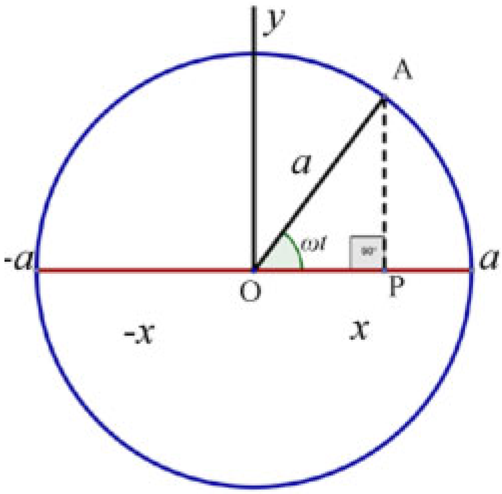
$\Rightarrow \quad \omega = \dfrac{v}{a} = \dfrac{15}{5} = 3$

period T is given by:

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{3} = 2.094$$

<u>Ans. period of motion is 2.1 secs.</u>

SHM and Circular Motion



The SHM-circle connection is used to solve problems concerning the time interval between particle positions.

To prove how SHM is derived from circular motion we must first draw a circle of radius 'a'(max. displacement).

Then, the projection(x-coord.) of a particle A is made on the diameter along the x-axis. This projection, as the particle moves around the circle, is the SHM displacement about O.

from triangle OAP

$$x = a\cos\omega t \qquad \text{(i}$$

$$\frac{dx}{dt} = -a\omega\sin\omega t$$

$$\frac{d^2x}{dt^2} = -a\omega^2\cos\omega t$$

but $a\cos\omega t = x$ from (i

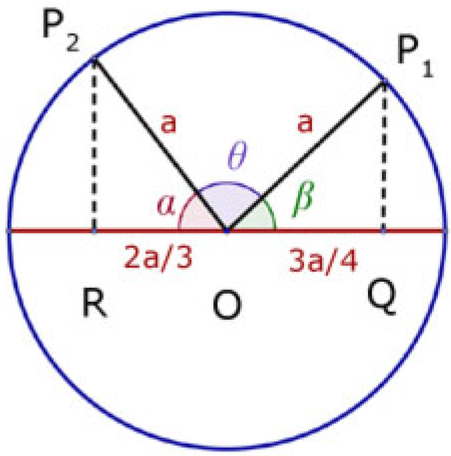
$$\Rightarrow \quad \underline{\frac{d^2x}{dt^2} = -\omega^2 x}$$

Q.E.D. the motion of particle P along the x-axis is S.H.M.

<u>Example</u>

A particle P moving with SHM about a centre O, has period T and amplitude a .

Q is a point $3a/4$ from O. R is a point $2a/3$ from O.What is the time interval(in terms of T) for P to move directly from Q to R? Answer to 2 d.p.



let the time interval between P_1 and P_2 be t secs.
let angle P_1OP_2 be θ rads.

period T is given by: $T = \dfrac{2\pi}{\omega}$, $\omega = \dfrac{2\pi}{T}$ (i

$\Rightarrow$ $t = \dfrac{\theta}{\omega}$ (ii

the angles in a straight line = 180 deg.
$\therefore$ from the diagram, $\pi = \theta + \alpha + \beta$
$\Rightarrow$ $\theta = \pi - \alpha - \beta$

from (ii substituting for θ

$$t = \frac{\pi - \alpha - \beta}{\omega}$$

from (i substituting for ω

$$t = \frac{T(\pi - \alpha - \beta)}{2\pi} \quad \text{(iii}$$

from the diagram $\cos\alpha = \dfrac{\frac{2a}{3}}{a} = \dfrac{2}{3}$

$\alpha = \cos^{-1}\left(\dfrac{2}{3}\right)$, $\alpha = 0.8412$

$\cos\beta = \dfrac{\frac{3a}{4}}{a} = \dfrac{3}{4}$

$\beta = \cos^{-1}\left(\dfrac{3}{4}\right)$, $\beta = 0.7227$

substituting for α and β in (iii

$$t = \frac{T(\pi - 0.8412 - 0.7227)}{2\pi}$$

$$= \frac{T}{2\pi}(1.5777) = 0.2510T$$

Ans. time interval is $0.25T$

Notes