A-LEVEL MATHS TUTOR

Mechanics

PART FIVE

STATICS

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Contents

Forces in Equilibrium on Particles

Below is a list of methods for describing forces in equilibrium acting on a particle. The forces act in **one** plane and are called **coplanar** .
In this context the word **equilibrium** means that the forces are in balance and there is no net force acting.

Vector Method

The condition for equilibrium is that the vector sum of forces(the resultant)is the **null vector**. This means that the multipliers for the **i**, **j** and **k** unit vectors are each zero.

Example #1

A particle at rest has forces (2**i** + 3**j**), (*m***i** + 6**j**) and (-4**i** + *n***j**) acting on it.
What are the values of *m* and *n* ?

For equilibrium, all the forces when added together(the vector sum) equal zero.

$$2\boldsymbol{i} + 3\boldsymbol{j} + m\boldsymbol{i} + 6\boldsymbol{j} - 4\boldsymbol{i} + n\boldsymbol{j} = 0$$
$$(2 + m - 4)\boldsymbol{i} + (3 + 6 + n)\boldsymbol{j} = 0$$

The scalar multipliers of each unit vector equal zero.

$$2 + m - 4 = 0$$
$$m = 4 - 2$$
$$\underline{m = 2}$$

$$3 + 6 + n = 0$$
$$n = -6 - 3$$
$$\underline{n = -9}$$

Ans. *m* = 2 and *n* = -9

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Example #2

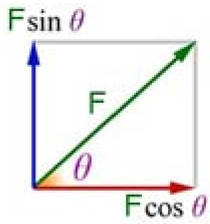
A particle at rest has an unknown force F and two other forces (2**i** - 5 **j**), (-3**i** + **j**), acting on it.
What is the magnitude of F ? (2 d.p.)
(all forces in Newtons, N)

for equilibrium the vector sum equals zero

$$2\boldsymbol{i} - 5\boldsymbol{j} - 3\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{F} = 0$$
$$\boldsymbol{F} = (2-3)\boldsymbol{i} + (-5+1)\boldsymbol{j}$$
$$\boldsymbol{F} = -\boldsymbol{i} - 4\boldsymbol{j}$$
$$|\boldsymbol{F}| = \sqrt{(-1)^2 + (-4)^2} = \sqrt{1+16} = \sqrt{17} = 4.12$$

Ans. force F = 4.12 N

Resolution Method



A force **F** can be replaced by two vectors that are at right angles to eachother, passing through the point of application. Hence if the angle between one component vector and the original vector is θ(theta), then the two components are **F**sinθ and **F**cosθ .

Problems are solved by resolving all the vectors into their horizontal and vertical components. The components are then resolved vertically and then horizontally to obtain two equation. These can be solved as simultaneous equations.

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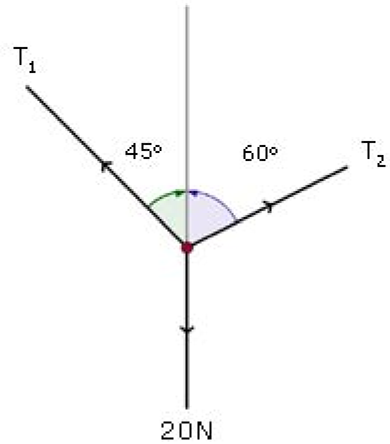
<u>Example</u>

A 2 kg mass is suspended by two light inextensible strings inclined at 60 deg. and 45 deg. to the vertical.
What are the tensions in the strings?(to 2 d.p.)
(assume g=10 ms^{-2})



resolving horizontally $\quad T_1 \sin 60^\circ = T_2 \sin 45^\circ$

$$T_1 = T_2 \frac{\sin 45^\circ}{\sin 60^\circ} = \frac{0.7071}{0.8660} T_2 = 0.8165 T_2$$

$$T_1 = 0.8165 T_2 \qquad \text{(i}$$

resolving vertically

$$T_1 \cos 60^\circ + T_2 \cos 45^\circ = 20 \qquad \text{(ii}$$

substituting for T_1 in (ii

$0.8165 T_2 (0.5) + 0.7071 T_2 = 20$

$0.4083 T + 0.7071 T_2 = 20$

$1.1154 T_2 = 20$

$T_2 = \dfrac{20}{1.1154} = 17.9308$

from (i $\quad T_1 = 0.8165 T_2$

$T_1 = 0.8165 \times 17.9308 = 14.6405$

<u>Ans. the tensions in the strings are 14.6 N and 17.9 N</u>

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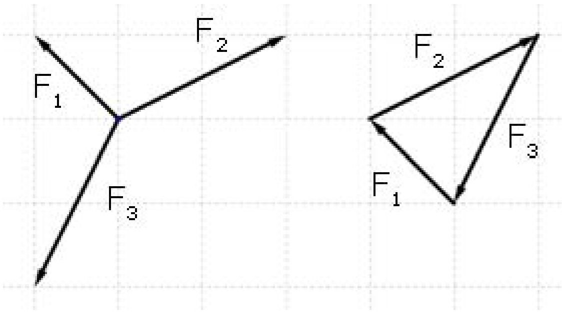
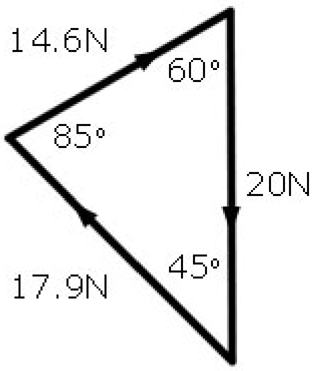
Triangle of Forces

When 3 coplanar forces acting at a point are in equilibrium, they can be represented in magnitude and direction by the adjacent sides of a triangle taken in order.



Example

Using the results from the previous example, the three forces acting on the 2 kg mass can be represented by a scale diagram.

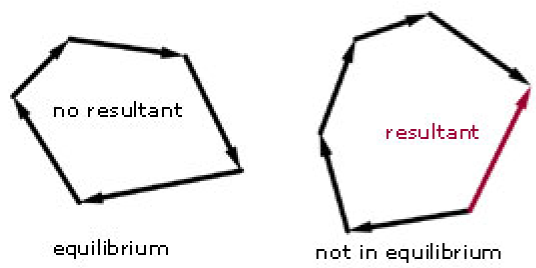


Our starting point is the 20N force acting downwards. One force acts at 45 deg. to this line and the other at 60 deg. So to find the magnitude of the two forces, draw lines at these angles at each end of the 20N force. Where the lines cross gives a vertex of the triangle. Measuring the lengths of the lines from this to the ends of the 20N force line will give the magnitudes of the required forces.

Polygon of Forces

For equilibrium, forces are represented in magnitude and direction to form a polygon shape.

If a number of forces are acting at a point, then the missing side in the polygon represents the resultant force. Note the arrow direction on this force is in the opposite direction to the rest.



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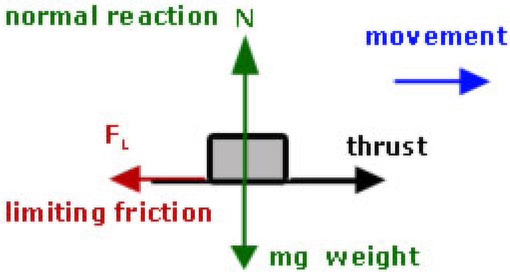
Friction

Friction is the force that opposes movement.



The 3 Laws of Friction

1.

The **limiting frictional force** (F_L)is directly proportional to the **normal contact force** (**N**)

$$F_L \propto N$$

note: limiting frictional force is the maximum frictional force

2.

The ratio of the **limiting frictional force** (F_L) to the **normal contact force** (**N**) is called the coefficient of friction (μ)

$$\mu = \frac{F_L}{N} \qquad F_L = \mu N$$

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3.

When there is no motion, but the object is on the point of moving,

applied force = frictional force(limiting friction)

and when there is motion,

applied force > frictional force(limiting friction)

then this equality applies:

$$F_L = \mu N$$

Up to this point, when the frictional force is **less** than limiting friction(maximum)*, then the inequality below applies.

$$F_L < \mu N$$

*object is static and not on the point of moving

Example #1

A flat stone is thrown horizontally across a frozen lake.
If the stone decelerates at 2.5 ms^{-2} , what is the coefficient of friction between the stone and the ice? (take g=10 ms^{-2})

since the stone is moving, the friction is limiting(max.)

$\therefore \quad F_L = \mu N \quad\quad$ (1

and vertically, $N = mg$

the stone is decelerating,

using Newton's Second law,

$$F_L = ma$$

substituting for F_L and N into (1

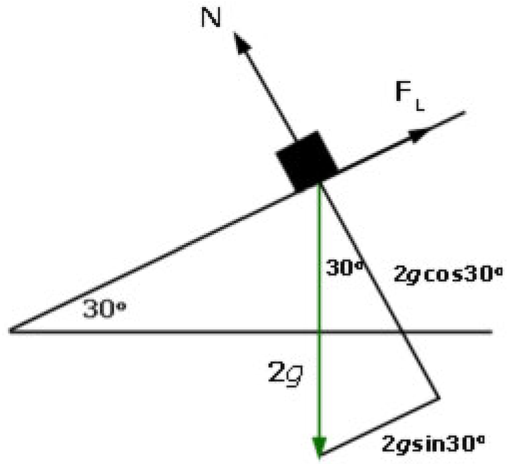
$$ma = \mu mg$$

$$\Rightarrow \quad \mu = \frac{ma}{mg} = \frac{a}{g} = \frac{2.5}{10} = 0.25$$

Ans. coefficient of friction between stone & ice is 0.25

Example #2

A 2 kg mass in limiting equilibrium rests on a rough plane inclined at an angle of 30 deg. to the horizontal.
Show that the coefficient of friction between the mass and the plane is √3 / 3 .



resolving along the plane

$$F_L = 2g\sin 30^\circ = 2g.\frac{1}{2}$$

$$F_L = g$$

resolving at right angles to the plane

$$N = 2g\cos 30^\circ$$

$$N = 2g\left(\frac{\sqrt{3}}{2}\right) = g\sqrt{3}$$

for limiting friction $\quad F_L = \mu N$

$$\Rightarrow \qquad \mu = \frac{F_L}{N} = \frac{g}{g\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

<u>Ans. coefft. of friction between 2 kg mass & plane is $\frac{\sqrt{3}}{3}$</u>

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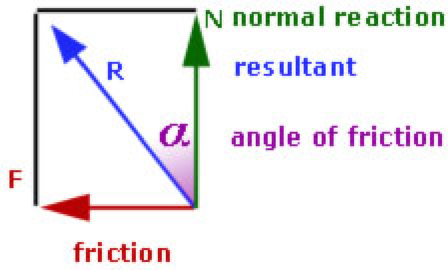
The Angle of Friction



If we examine the normal reaction force($\boldsymbol{N}$) and the frictional force($\boldsymbol{F_L}$) when it is limiting, then the equation $\boldsymbol{F_L} = \mu \boldsymbol{N}$ applies.

If the resultant between $\boldsymbol{N}$ and $\boldsymbol{F_L}$ is $\boldsymbol{R}$, and it is inclined at an angle α(alpha) to the normal $\boldsymbol{N}$, then we can write equations for $\boldsymbol{F_L}$ and $\boldsymbol{N}$ in terms of $\boldsymbol{R}$.

$$\frac{F_L}{N} = \mu$$

$$F_L = R\sin\alpha$$

$$N = R\cos\alpha$$

$$\frac{F_L}{N} = \frac{R\sin\alpha}{R\cos\alpha} = \tan\alpha$$

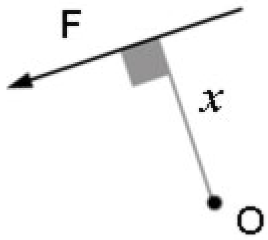
$$\Rightarrow \quad \underline{\mu = \tan\alpha}$$

In example #2(above) the angle of friction = tan $^{-1}$(√3 / 3)
= tan $^{-1}$(0.5773) = 30° (the angle of the plane)

Rigid Bodies

The Moment of a Force

The moment M (turning effect) of a force about a point O is the product of the magnitude of the force (F) and the perp. distance (x)to the point of application.



$$M = F\,x$$

By convention, **anti-clockwise** moments are **positive**.

The Principle of Moments

For a rigid body acted upon by a system of coplanar forces, equilibrium is achieved when:

i) the vector sum of the coplanar forces = 0

ii) there is no net turning effect produced by the forces
(the sum of clockwise & anti-clockwise moments = 0)

Parallel forces acting on a beam

When attempting problems concerning a balance points or fulcrum, remember that there is always an upward force acting.

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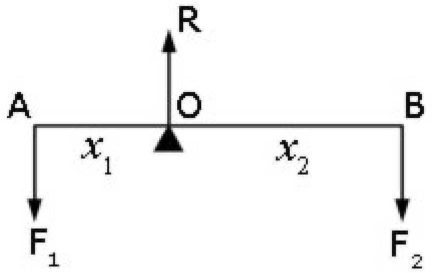
Example #1   fulcrum near centre



Consider two forces F_1 and F_2 acting vertically downwards at either end of a beam of negligible mass.

When the beam is balanced at its fulcrum O(i.e. horizontal), the sum of the downward forces equals the sum of the upward forces.

If the reaction force at the fulcrum is R,

$$R = F_1 + F_2$$

taking moments about O,

$$F_1 x_1 - F_2 x_2 = 0$$

$$F_1 x_1 = F_2 x_2$$

$$\frac{F_1}{F_2} = \frac{x_2}{x_1}$$

taking moments about A,

$$(F_1 + F_2)x_1 - F_2(x_1 + x_2) = 0$$

$$(F_1 + F_2)x_1 = F_2(x_1 + x_2)$$

$$\underline{\frac{x_1}{x_1 + x_2} = \frac{F_2}{F_1 + F_2}}$$

taking moments about B,

$$F_1(x_1 + x_2) - (F_1 + F_2)x_2 = 0$$

$$F_1(x_1 + x_2) = (F_1 + F_2)x_2$$

$$\underline{\frac{x_2}{x_1 + x_2} = \frac{F_1}{F_1 + F_2}}$$

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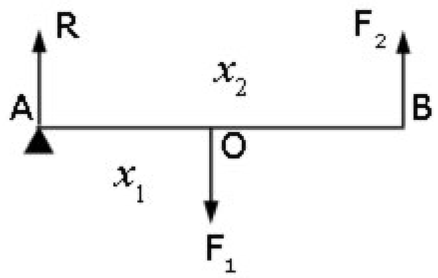
Example #2 fulcrum at one end



Consider two forces F_1 and F_2 acting on a beam of negligible mass. One force acts vertically downwards near the centre, while the other acts vertically upwards at the end.

When the beam is balanced, the sum of the downward forces equals the sum of the upward forces.

If the reaction force at the fulcrum is R,

$$R = F_2 - F_1$$

taking moments about O,

$$F_2(x_2 - x_1) - (F_2 - F_1)x_1 = 0$$

$$F_2(x_2 - x_1) = (F_2 - F_1)x_1$$

$$\underline{\frac{x_1}{x_2 - x_1} = \frac{F_2}{F_2 - F_1}}$$

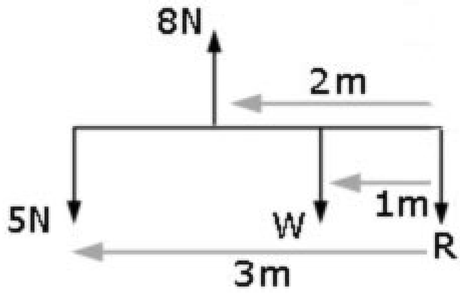
taking moments about B,

$$F_1(x_2 - x_1) - (F_2 - F_1)x_2 = 0$$

$$F_1(x_2 - x_1) = (F_2 - F_1)x_2$$

$$\underline{\frac{x_2}{x_2 - x_1} = \frac{F_1}{F_2 - F_1}}$$

Example #3 a typical problem



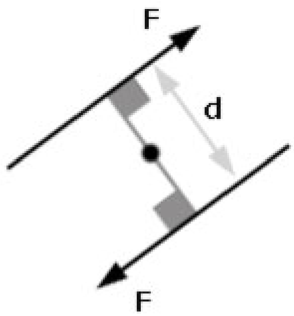
A beam of negligible weight is horizontal in equilibrium, with forces acting upon it, as shown in the diagram.
Calculate the value of the weights R and W.

vertically $$8 = 5 + W + R$$
$$R = 3 - W \qquad (i$$

taking moments about R
$$(5 \times 3) + (W \times 1) = 8 \times 2$$
$$15 + W = 16$$
$$W = 16 - 15$$
$$\underline{W = 1N}$$

from (i
$$R = 3 - 1$$
$$\underline{R = 2N}$$

Couples



The turning effect of two equal and opposite parallel forces acting about a point equals the product of one force(F) and the perpendicular distance between the forces(d).

couple = F x d

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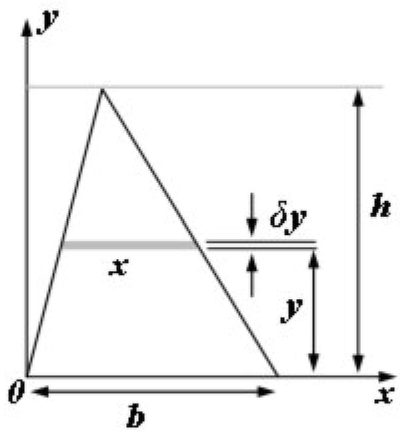
Centre of Mass

To understand the concept of centre of mass we must first consider a rigid body to be made of an infinite number of particles. Each particle has a gravitational force on it directed towards the centre of the Earth. Assuming these forces are parallel, the weight of the body equals the sum of all the tiny particle weights.

This effective weight may be considered to act through a particular point called the ***centre of mass***(sometimes termed the **centroid**)

When trying to calculate the centre of mass of a uniform solid eg a cone or hemishpere, we consider the whole mass and its moment about an axis and equate this to the sum of all the moments of constituent masses about the axis. The constituent masses often take the form of thin slices of the regular solid.

Centre of mass of a triangular lamina



Consider a small horizontal strip of area δA(delta A).

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area of small strip $\delta A = x\delta y$

by similar triangles,

$$\frac{x}{(h-y)} = \frac{b}{h}$$

$$x\delta y = \left[\frac{b(h-y)}{h}\right]\delta y$$

total area A is given by,

$$A = \frac{1}{2}bh$$

let distance of centroid from x-axis be $y_{\mathbf{c}}$

taking moments about the x-axis

area of triangle $\times$ $y_{\mathbf{c}}$ = sum of slices $\times$ slice displacements

$$\frac{1}{2}bhy_{\mathbf{c}} = \int_0^h \left[\frac{b(h-y)}{h}\right] y dy$$

$$y_{\mathbf{c}} = \frac{b}{h}\cdot\frac{2}{bh}\int_0^h (h-y)y dy$$

$$= \frac{2}{h^2}\int_0^h (hy - y^2) dy$$

$$= \frac{2}{h^2}\left[\frac{hy^2}{2} - \frac{y^3}{3}\right]_0^h$$

$$= \frac{2}{h^2}\left[\frac{h^3}{2} - \frac{h^3}{3}\right] - \frac{2}{h^2}[0]$$

$$= \frac{2}{h^2}\left[\frac{h^3}{6}\right]$$

$$\underline{y_{\mathbf{c}} = \frac{h}{3}}$$

The centre of mass is on a horizontal line a third of the height up from the base

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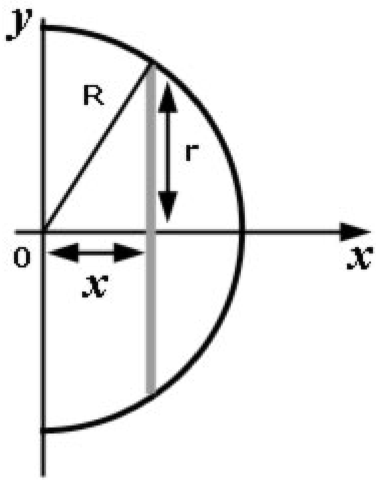
Centre of mass of a hemisphere



The method of calculation for the centre of mass of a 3D object is very similar to that of a 2D object. In this case we sum thin 2D slices.
The hemispherical axis is the x-axis and this time we consider circular slices of thickness δx.

The radius r of each slice is given by Pythagoras' Theorem:

$$R^2 = r^2 + x^2$$

hence the area of a slice is given by,

$$\Pi r^2 = \Pi(R^2 - x^2)$$

so the volume of a slice δV is given by:

$$\delta V = \Pi(R^2 - x^2)\delta x$$

Let the centroid be at position x_C from the origin O.

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$$\frac{2}{3}\pi R^3 x_{\mathbf{c}} = \int_0^R \pi(R^2 - x^2)\,x dx$$

$$x_{\mathbf{c}} = \frac{1}{\frac{2}{3}\pi R^3}\int_0^R \pi(R^2 - x^2)\,x dx$$

$$= \frac{1}{\frac{2}{3}R^3}\int_0^R (R^2 x - x^3)dx$$

$$= \frac{1}{\frac{2}{3}R^3}\left[R^2\frac{x^2}{2} - \frac{x^4}{4}\right]_0^R$$

$$= \frac{1}{\frac{2}{3}R^3}\left[R^2\frac{R^2}{2} - \frac{R^4}{4}\right] - \frac{1}{\frac{2}{3}R^3}[0]$$

$$= \frac{1}{\frac{2}{3}R^3}\left[\frac{R^4}{2} - \frac{R^4}{4}\right]$$

$$= \frac{1}{\frac{2}{3}R^3}\left[\frac{R^4}{4}\right] = \frac{3}{2R^3}\left[\frac{R^4}{4}\right]$$

$$\underline{x_{\mathbf{c}} = \frac{3}{8}R}$$

<u>Toppling</u>

This is to do with a rigid body in contact with a rough plane. Normally the body sits on the plane in equilibrium, with the plane horizontal.

However if the plane is tilted, this state of affairs only remains until what is called the 'tipping point'. This is when a **vertical line** passing through the **centre of gravity** of the body is on the point of **falling outside** of the body's **base area**.

When the line falls outside of the base area, the body tips over.

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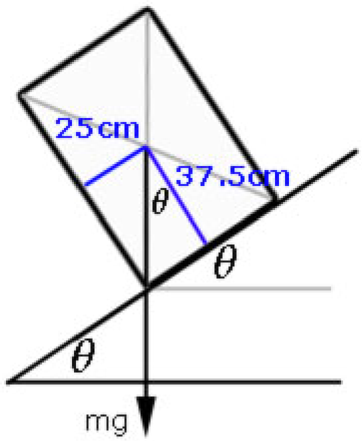
<u>Example</u>

A uniform cuboidal block of dimensions, height 75cm and base 30cm x 50cm rests on a rough plane, with its 50cm side up the plane.
Calculate the angle of inclination of the plane when the block is on the point of toppling(ans. to 1 d.p.)
(assume that the frictional forces on the base of the block are too high for the block to slide)



$\tan\Theta = 25/37.5$

<u>$\Theta = 33.7^{\circ}$</u>

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