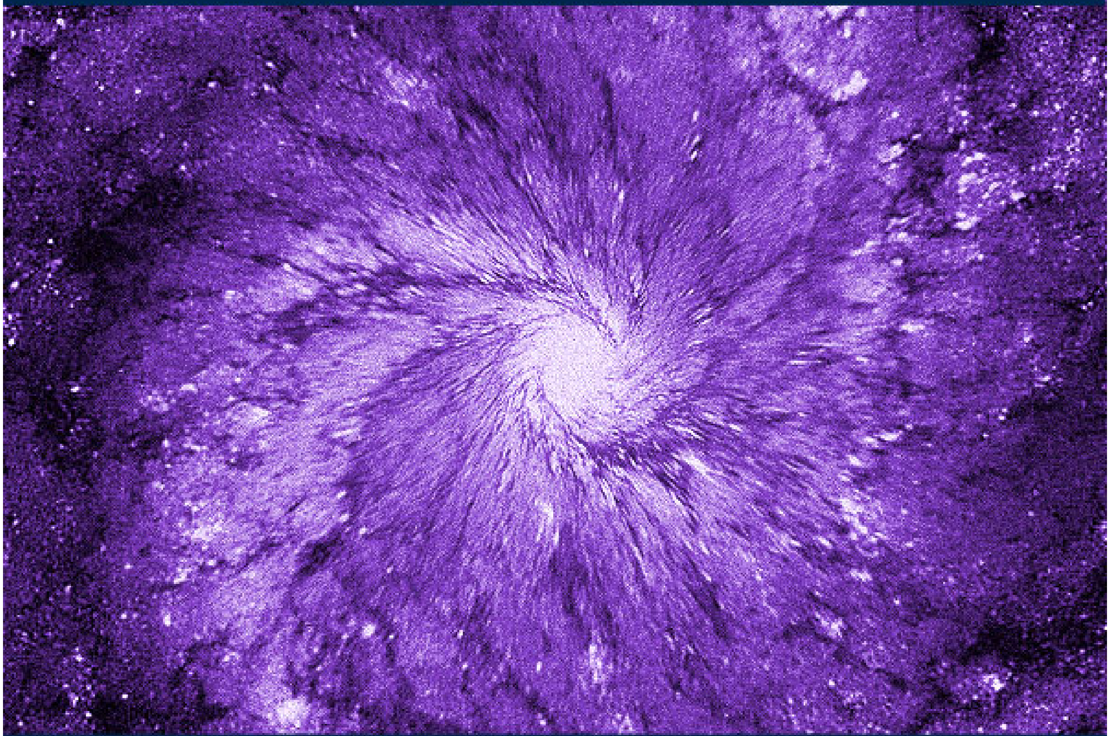
A-LEVEL MATHS TUTOR

Pure Maths



PART TWO

INTEGRATION

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The Integration Formula

The Integration Formula

The expression to be integrated is the derivative of some function eg f(x) called the **integrand**.

When this expression is integrated the original function is restored plus a constant (C) called the constant of integration.

This is called the **indefinite integral** when the integration is not between two limiting values of x.

However when the integration is between two limiting values of x then the integral is called the **definite integral** and the constant of integration is not involved.

For any variable 'x' to the power of 'n' the integral is given by:

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

In other words, increase the power of x by '1' and divide x by the new index.

Rule #1

Any constant(eg C) multiplied by a function f(x) can be integrated by placing the constant before the integration sign.

$$\int Cf(x)dx = C\int f(x)dx$$

Example

$$\int 5\sin(x)dx = 5\int \sin(x)dx = -5\cos(x)$$

Rule #2

The integral of two separate functions which are added together is the same as each function integrated separately then added together.

$$\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$$

Example

$$\begin{aligned}&\int (5x^2 + 4e^x)dx \\ &= 5\int x^2 dx + 4\int e^x dx \\ &= \frac{5}{3}x^3 + 4e^x\end{aligned}$$

Rule #3

The addition of a constant to a variable doesn't change the form of the integral. However, x must be in the first degree ie no higher powers of x are involved. ('a' is a constant)

$$\text{if} \quad \int f(x)dx = F(x) \quad \text{then} \quad \int f(x+a)dx = F(x+a)$$

Example

$$\int x^3 dx = \frac{x^4}{4} \quad \Rightarrow \quad \int (x+5)^3 dx = \frac{(x+5)^4}{4}$$

Rule #4

If 'a' & 'b' are constants then x can be replaced by 'bx+a' with the integral remaining in the same form.

$$\text{if } \int f(x)dx = F(x)$$

$$\text{then } \int f(bx+a)dx = \frac{1}{b}F(bx+a)$$

Example

$$\int x dx = \frac{x^2}{2}$$

$$\int (3x+4)dx = \frac{1}{3}\frac{(3x+4)^2}{2}$$

$$= \frac{(3x+4)^2}{6}$$

Integration by Substitution

The Substitution method(or 'changing the variable')

This is best explained with an example:

$$\int (2x+5)^3\,dx$$

Like the Chain Rule simply make one part of the function equal to a variable eg u,v, t etc.

$$\text{let} \qquad t = (2x+5)$$

Differentiate the equation with respect to the chosen variable.

$$\frac{dt}{dx} = 2$$

Rearrange the substitution equation to make 'dx' the subject.

$$dx = \frac{dt}{2}$$

Substitute for 'dx' into the original expression.

$$\int (2x+5)^3 \frac{dt}{2}$$

Substitute the chosen variable into the original function.

$$\int t^3 . \frac{dt}{2}$$

Integrate with respect to the chosen variable.

$$\int \frac{t^3}{2} dt = \frac{t^4}{2.4} = \frac{t^4}{8} + C$$

Restate the original expression and substitute for t.

$$\int (2x+5)^3 dx = \frac{(2x+5)^4}{8} + C$$

NB Don't forget to add the Constant of Integration(C) at the end. Remember this is an indefinite integral.

Example #1

$$\int 5xe^{x^2+1}dx$$

make $\quad t = x^2 + 1$

then $\quad \dfrac{dt}{dx} = 2x$

$$xdx = \frac{dt}{2}$$

$$\int 5xe^{x^2+1}dx = \int 5e^t.\frac{dt}{2}$$

$$= \int \frac{5}{2}e^t dt$$

$$= \frac{5}{2}e^t + C$$

$$= \frac{5}{2}e^{x^2+1} + C$$

Example #2

$$\int \frac{1}{x}\ln x dx$$

make $\quad t = \ln x$

then $\quad \dfrac{dt}{dx} = \dfrac{1}{x}$

$$xdt = dx$$

$$dt = \frac{dx}{x}$$

$$\int \frac{1}{x}\ln x dx = \int t dt$$

$$= \frac{t^2}{2}$$

$$= \frac{1}{2}(\ln x)$$

Example #3

$$\int \sin^3 x \cos x dx$$

let $\quad t = \sin x$

then $\quad \frac{dt}{dx} = \cos x$

$$dt = \cos x dx$$

$$\int \sin^3 x \cos x dx = \int t^3 dt$$

$$= \frac{t^4}{4} + C$$

$$= \frac{\sin^4 x}{4} + C$$

Integration 'by parts'

Integration 'By Parts' - from the Product Rule

The integration of expressions where there are two separate functions multiplied together, is essentially by an amended version of Leibnitz's Product Rule.

to integrate an expression of the type

$$\int f(x).g(x)dx$$

make $\quad u = f(x) \quad$ and $\quad v = g(x)$

the expression becomes $\int uv dx$

using the Product Rule:

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\Rightarrow \qquad u\frac{dv}{dx} = \frac{d(uv)}{dx} - v\frac{du}{dx}$$

integrating with respect to x

$$\int u\left(\frac{dv}{dx}\right)dx = uv - \int v\left(\frac{du}{dx}\right)dx$$

$$\int u dv = uv - \int v du$$

Example #1

integrate $\int 2xe^{3x}dx$

make $u = 2x$ and $\frac{dv}{dx} = e^{3x}$

$dv = e^{3x}dx$

then $\frac{du}{dx} = 2$ and $v = \frac{1}{3}e^{3x}$

$du = 2dx$

using the formula

$$\int udv = uv - \int vdu$$

$$\int 2xe^{3x}dx = 2x.\frac{1}{3}e^{3x} - \int \frac{1}{3}e^{3x}.2dx$$

$$= \frac{2x}{3}e^{3x} - \frac{2}{3}.\frac{1}{3}e^{3x}$$

$$= \frac{2x}{3}e^{3x} - \frac{2}{9}e^{3x} + C$$

Example #2

integrate $\int 2x\cos x dx$

make $u = 2x$ and $\frac{dv}{dx} = \cos x$

$dv = \cos x dx$

then $\frac{du}{dx} = 2$ and $v = \sin x$

$du = 2dx$

using the formula

$$\int udv = uv - \int vdu$$

$$\int 2x\cos x dx = 2x.\sin x - \int \sin x.2dx$$

$$= 2x\sin x - 2(-\cos x) + C$$

$$= 2x\sin x + 2\cos x + C$$

Example #3

integrate $\int 3x \ln x dx$

make $u = \ln x$ and $\frac{dv}{dx} = 3x$

$dv = 3xdx$

then $\frac{du}{dx} = \frac{1}{x}$ and $v = \frac{3}{2}x^2$

$du = \frac{dx}{x}$

using the formula

$$\int u dv = uv - \int v du$$

$$\int 3x \ln x dx = \ln x . \frac{3}{2}x^2 - \int \frac{3}{2}x^2 . \frac{dx}{x}$$

$$= \frac{3}{2}x^2 \ln x - \int \frac{3}{2} x dx$$

$$= \frac{3}{2}x^2 \ln x - \frac{3}{2} . \frac{x^2}{2}$$

$$= \frac{3}{2}x^2 \ln x - \frac{3x^2}{4}$$

Integration of Algebraic Fractions

Before beginning this topic it is advised that you read and understand 'partial fractions' in the **algebra** section first. The examples given here assume knowledge of this.

Denominator 1st degree(x)

Example

$$\text{find} \quad \int \frac{3}{5x+2}dx$$

$$\text{using} \int \frac{1}{ax+b} = \frac{1}{a}\ln|ax+b|$$

$$\Rightarrow \quad \int \frac{3}{5x+2}dx = 3.\frac{1}{5}\ln|5x+2|$$

$$\int \frac{3}{5x+2}dx = \frac{3}{5}\ln|5x+2|$$

Denominator 2nd degree(x^2)

Example

find $\int \frac{2x-2}{(x+1)^2}dx$

using partial fractions

$$\frac{2x-2}{(x+1)^2} \equiv \frac{A}{x+1} + \frac{B}{(x+1)^2}$$

$$\equiv \frac{A(x+1)+B}{(x+1)^2}$$

$$2x-2 = A(x+1)+B$$

making $x=-1$

$$-2-2 = \quad 0 \quad +B$$

$$\underline{B=-4}$$

making $x=2$

$$4-2 = A(3)-4$$

$$2 = 3A-4, \quad 3A=6, \quad \underline{A=2}$$

$$\Rightarrow \qquad \frac{2x-2}{(x+1)^2} \equiv \frac{2}{x+1} - \frac{4}{(x+1)^2}$$

$$\int \frac{2x-2}{(x+1)^2}dx = \int \left(\frac{2}{x+1} - \frac{4}{(x+1)^2} \right) dx$$

$$\int \frac{2x-2}{(x+1)^2}dx = 2\ln(x+1) + 4(x+1)^{-1}$$

Denominator 3rd degree(x^3)

Example

find $\int \frac{5+x}{(1-x)(5+x^2)}dx$

using partial fractions

$$\frac{5+x}{(1-x)(5+x^2)} \equiv \frac{A}{x+1} + \frac{Bx+C}{5+x^2}$$

$$\equiv \frac{A(5+x^2)+(Bx+C)(1-x)}{(1-x)(5+x^2)}$$

$$5+x = A(5+x^2)+(Bx+C)(1-x)$$

$$5+x = 5A + Ax^2 + Bx + 1 - Bx^2 - cx \quad *$$

making $x = 1$

$$5+1 = A(5+1) + \quad 0$$

$$6 = 6A, \qquad \underline{A = 1}$$

making $x = 0$

$$5 = 5A + C, \qquad 5 = 5 + C, \qquad \underline{C = 0}$$

equating coefficients of x^2 *

$$0 = A - B, \quad \underline{B = A = 1}$$

$$\Rightarrow \quad \frac{5+x}{(1-x)(5+x^2)} \equiv \frac{1}{1-x} + \frac{x}{5+x^2}$$

$$\int \frac{5+x}{(1-x)(5+x^2)}dx = \int\left(\frac{1}{1-x} - \frac{x}{5+x^2}\right)dx$$

$$\int \frac{5+x}{(1-x)(5+x^2)}dx = -\ln(x-1) + \frac{1}{2}\ln(5+x^2)$$

Definite Integrals

The 'definite Integral' equation

If a function F(x) is the integral of the function f(x)

$$F(x) = \int f(x)dx$$

then an integral of the form:

$$\int_a^b f(x)dx = \left[F(x) \right]_a^b = F(b) - F(a)$$

is known as the **definite integral**, where a, b are called the **limits** of the integral.

Example #1

$$\text{evaluate } \int_1^2 (2x^2 + 1)dx$$

$$\int_1^2 (2x^2 + 1)dx = \left[\frac{2x^3}{3} + x \right]_1^2$$

$$= \left[\frac{2.2^3}{3} + 2 \right] - \left[\frac{2.1^3}{3} + 1 \right]$$

$$= \left[\frac{2.8}{3} + 2 \right] - \left[\frac{2}{3} + 1 \right]$$

$$= \frac{16}{3} + 2 - \frac{2}{3} + 1$$

$$= \frac{14}{3} + 3 = 4\tfrac{2}{3} + 3 = 7\tfrac{2}{3}$$

$$\underline{\int_1^2 (2x^2 + 1)dx = 7\tfrac{2}{3}}$$

Example #2

evaluate $\int_2^3 (x^3 + 2x)dx$

$$\int_2^3 (x^3 + 2x)dx = \left[\frac{x^4}{4} + x^2\right]_2^3$$
$$= \left[\frac{3^4}{4} + 3^2\right] - \left[\frac{2^4}{4} + 2^2\right]$$
$$= \left[\frac{81}{4} + 9\right] - \left[\frac{16}{4} + 4\right]$$
$$= 20\tfrac{1}{4} + 9 - 4 - 4$$
$$= 21\tfrac{1}{4}$$

$$\underline{\int_2^3 (x^3 + 2x)dx = 21\tfrac{1}{4}}$$

Example #3

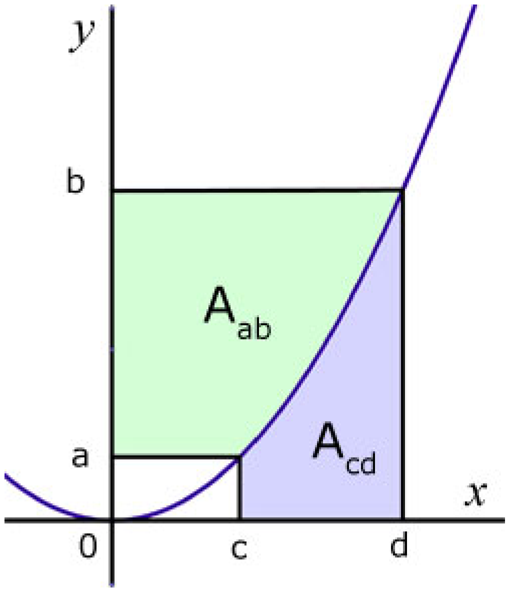
evaluate $\int_0^2 (3x^2 + 2x + 5)dx$

$$\int_0^2 (3x^2 + 2x + 5)dx = \left[\frac{3x^3}{3} + \frac{2x^2}{2} + 5x\right]_0^2$$
$$= \left[2^3 + 2^2 + 5.2^1\right] - \left[0\right]$$
$$= \left[8 + 4 + 10\right]$$
$$= 22$$

$$\underline{\int_0^2 (3x^2 + 2x + 5)dx = 22}$$

The Area Under a Curve

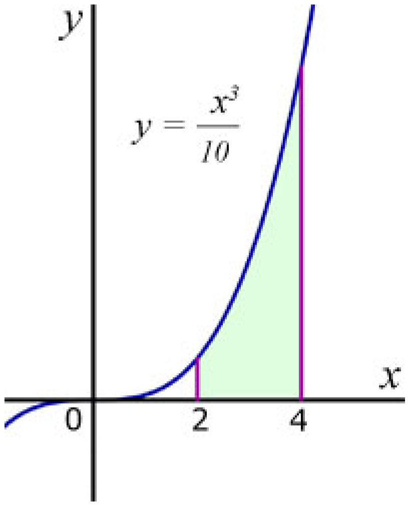
Area under a curve related to different axes



$$A_{ab} = \int_a^b x\,dy \qquad A_{cd} = \int_c^d y\,dx$$

Example #1

Find the area 'A' enclosed by the x-axis, x=2, x=4 and the graph of $y=x^3/10$.



$$A = \int_2^4 y\,dx$$

$$= \int_2^4 \frac{x^3}{10}\,dx$$

$$= \left[\frac{x^4}{4.10}\right]_2^4 = \left[\frac{x^4}{40}\right]_2^4$$

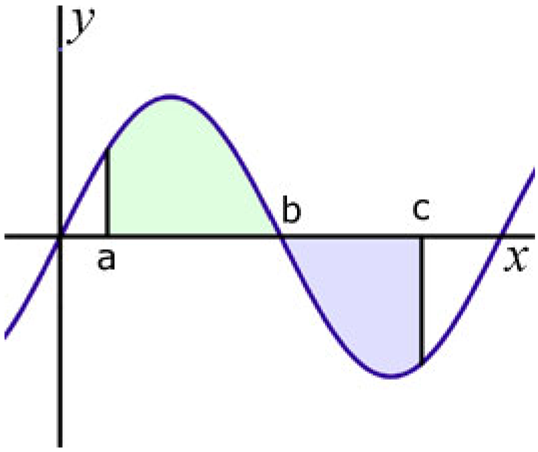
$$= \left[\frac{4^4}{40}\right] - \left[\frac{2^4}{40}\right]$$

$$= \left[\frac{256}{40}\right] - \left[\frac{16}{40}\right]$$

$$= \left[\frac{240}{40}\right] = [6]$$

area 'A' is 6

Positive and negative area



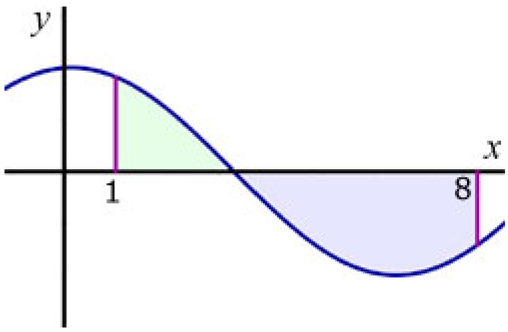
$$A_{ac} = \int_a^b y\,dx - \int_b^c y\,dx$$
$$= \int_a^b f(x)dx + \left| \int_b^c f(x)dx \right|$$

note: This expression calculates the **absolute area** between the curve the vertical lines at 'a' and 'b' and the x-axis. It takes no account of sign. If sign were an issue then the two integrals on the first line would be subtracted and not added.

Unless told differently, assume that the absolute area is required.

Example #1

Find the area 'A' enclosed by the x-axis, x=1, x=8 and the graph of y=2sin[(x+3)/2].



The curve crosses the x-axis at y=0.

Therefore 2sin[(x+3)/2]=0

Sine is zero when the angle is 0,180 or 360 deg.

(zero, pi and 2 pi)

$$\frac{x+3}{2} = \pi, \quad x+3 = 2\pi, \quad x = 2\pi - 3, \quad x = 3.28$$

$$A = \int_{1}^{3.28} y\,dx + \left| \int_{3.28}^{4} y\,dx \right|$$

$$= \int_{1}^{3.28} 2\sin[(x+3)/2]dx + \left| \int_{3.28}^{8} 2\sin[(x+3)/2]dx \right|$$

$$A = \left[-2\cos\left(\frac{x+3}{2}\right)\right]_1^{3.28} + \left|\left[-2\cos\left(\frac{x+3}{2}\right)\right]_{3.28}^{8}\right|$$

$$= \left[-2\cos\left(\frac{3.28+3}{2}\right)\right] - \left[-2\cos\left(\frac{1+3}{2}\right)\right]$$
$$+ \left|\left[-2\cos\left(\frac{8+3}{2}\right)\right] - \left[-2\cos\left(\frac{3.28+3}{2}\right)\right]\right|$$

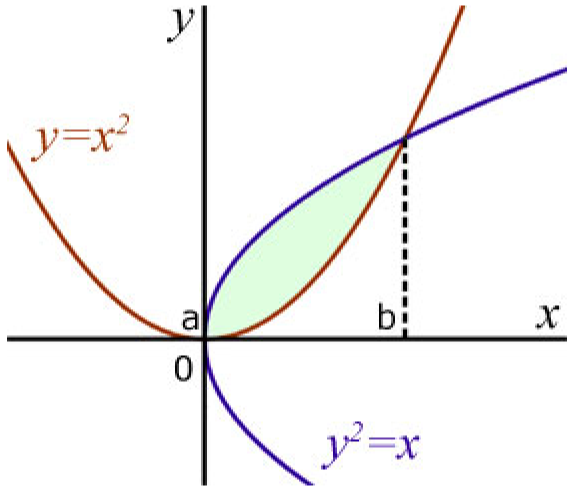
$$= \left[-2\cos(3.14)\right] - \left[-2\cos(2)\right]$$
$$+ \left|\left[-2\cos(5.5)\right] - \left[-2\cos(3.14)\right]\right|$$

$$= \left[-2(1)\right] - \left[-2(-0.416)\right]$$
$$+ \left|\left[-2(0.709)\right] - \left[-2(-1)\right]\right|$$

$$= 2 - 0.83 + \left|(-1.42 - 2)\right|$$
$$= 1.17 + 3.42$$
$$= 4.59$$

<u>area $A = 4.59$</u>

Area bounded by two curves

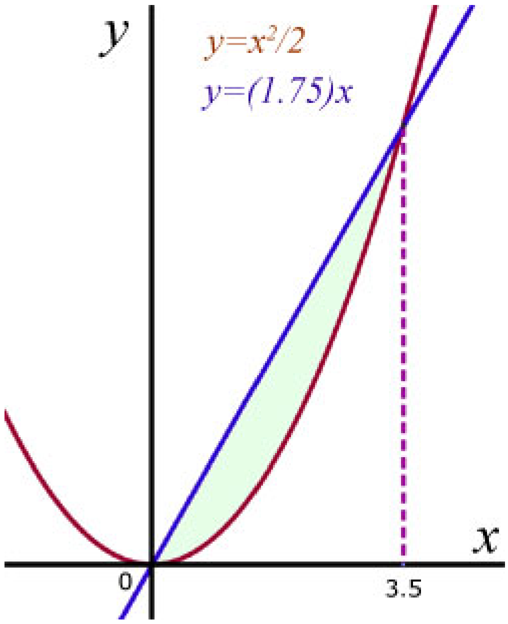


$$A_{ab} = \int_a^b y_1 dx - \int_a^b y_2 dx$$

<u>Example</u>

To 3 d.p. calculate the area 'A' included between the curves $y=x^2/2$ and $y=(0.75)x$

first find the x value where the curves cross



$$y_1 = \frac{x^2}{2}, \qquad y_2 = \frac{7x}{4}$$

where the curves cross $\quad y_1 = y_2$

$$\frac{x^2}{2} = \frac{7x}{4}, \quad \Rightarrow \quad 4x^2 = 14x$$

$$\Rightarrow \quad 2x = 7, \quad \therefore \quad \underline{x = 3.5}$$

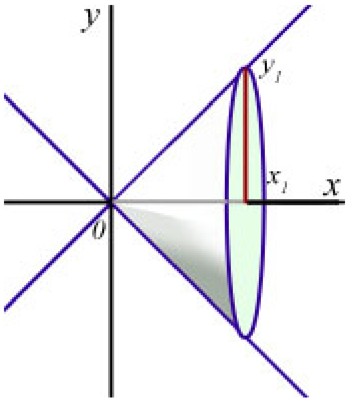
The area 'A' is the difference between the area under the straight line and the area under the parabola, from x=0 to x=3.5 .

$$A = \int_0^{3.5} \frac{7}{4}x\,dx - \int_0^{3.5} \frac{x^2}{2}dx$$
$$= \left[\frac{7}{4}.\frac{x^2}{2}\right]_0^{3.5} - \left[\frac{1}{2}.\frac{x^3}{3}\right]_0^{3.5}$$
$$= \left[\frac{7}{8}.(3.5)^2\right] - \left[\frac{(3.5)^3}{6}\right]$$
$$= \frac{7}{8}(12.25) - \frac{1}{6}(42.875)$$
$$= 10.719 - 7.146$$
$$= 3.573$$

$\underline{\text{area} \quad A = 3.573}$

Volumes of Revolution

Method



A volume(rotated around the x-axis)is calculated by first considering a particular value of a function, y_1, up from a value of x at x_1 . The line x_1y_1 may be considered as the 'radius' of the solid at that particular value of x.

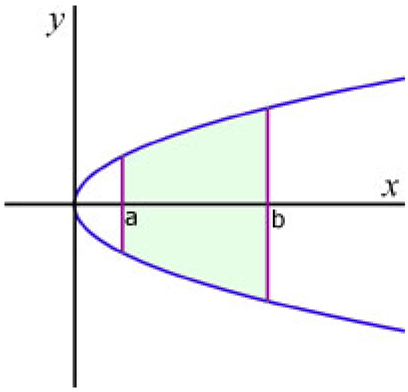
If you were to square the y-value and multiply it by pi, then a cross-sectional area would be created.

Making a solid of revolution is simply the method of summing all the cross-sectional areas along the x-axis between two values of x.

(compare: area of a cylinder = cross-sectional area x length)

The method for solids rotated around the y-axis is similar.

Rotation around the x-axis

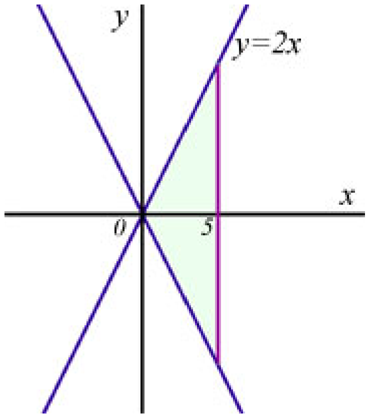


The volume $V_x$of a curve y=f(x) rotated around the x-axis between the values of x of a and b, is given by:

$$V_x = \pi \int_a^b y^2 dx$$

Example

What is the volume V of the cone swept out by the line y=2x rotated about the x-axis between x=0 and x=5?



using $V_x = \pi \int_a^b y^2 dx$

and $y = 2x$

$$\Rightarrow \quad V_{0,5} = \pi \int_0^5 (2x)^2 dx$$

$$= \pi \int_0^5 4x^2 dx$$

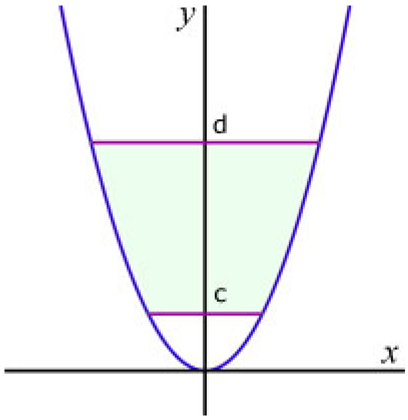
$$= \pi \left[\frac{4}{3}(x^3) \right]_0^5$$

$$= \pi \left[\frac{4}{3}(5^3) \right] - \pi [0]$$

$$= \pi \frac{4(125)}{3} = \pi \frac{500}{3} = 166.\dot{6}\,\pi$$

<u>volume V of cone is $166.\dot{6}$</u> π

Rotation around the y-axis

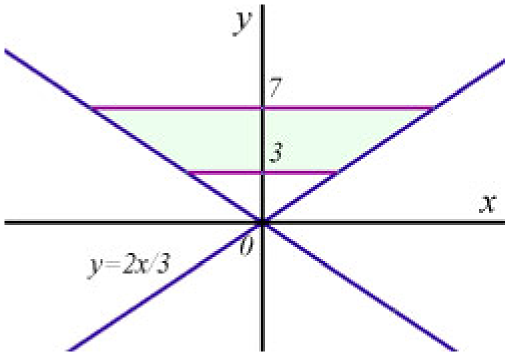


The volume V_y of a curve y=f(x) rotated around the x-axis between the values of y of c and d, is given by:

$$V_y = \pi \int_c^d x^2 dy$$

Example

What is the volume V of the 'frustrum'(cone with smaller cone-shape removed) produced when the line y=2x/3 is rotated around the y-axis, when the centres of the upper and lower areas of the frustrum are at 0,7 and 0,3



using $V_y = \pi \int_c^d x^2 dy$

and $y = \frac{2x}{3}, \quad \therefore \quad x = \frac{3y}{2}$

$$\Rightarrow \quad V_{3,7} = \pi \int_3^7 \left(\frac{3y}{2}\right)^2 dy$$

$$= \pi \int_3^7 \frac{9}{4} y^2 dy$$

$$= \pi \left[\frac{9}{4}\frac{y^3}{3}\right]_3^7$$

$$= \pi \left[\frac{9}{4}\frac{7^3}{3}\right] - \pi \left[\frac{9}{4}\frac{3^3}{3}\right]$$

$$= \pi \left[\frac{3}{4}(343)\right] - \pi \left[\frac{3}{4}(27)\right]$$

$$= \pi \left[\frac{1029}{4}\right] - \pi \left[\frac{81}{4}\right]$$

$$= \frac{948}{4}\pi = 237\pi$$

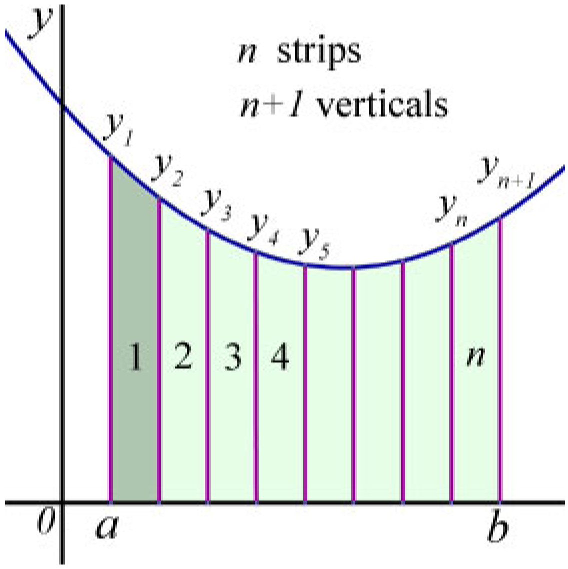
volume of cone is 237π

The Trapezium(Trapezoid) Rule

Theory & method

The Trapezium Rule is a method of finding the approximate value of an integral between two limits.

The area involved is divided up into a number of parallel strips of equal width.



Each area is considered to be a trapezium(trapezoid).

If there are **n** vertical strips then there are **n+1** vertical lines(ordinates) bounding them.

The limits of the integral are between **a** and **b**, and each vertical line has length y_1 y_2 y_3... y_{n+1}

$$\text{width of each strip } = \frac{(b-a)}{n}$$

$$\text{area of first strip}(n=1)$$
$$= (\text{width of strip}) \times (\text{average length of verticals})$$
$$= \frac{(b-a)}{n}\left(\frac{y_2+y_1}{2}\right)$$

Therefore in terms of the all the vertical strips, the integral is given by:

$$\int_a^b f(x)dx$$
$$\approx\left(\frac{b-a}{n}\right)\left[\frac{1}{2}(y_1+y_2)+\frac{1}{2}(y_2+y_3)+\frac{1}{2}(y_3+y_4)+..\frac{1}{2}(y_n+y_{n+1})\right]$$
$$\approx\left(\frac{b-a}{n}\right)\left[\frac{1}{2}(y_1+y_{n+1})+(y_2+y_3+y_4+\ldots+y_n)\right]$$

approx. integral = (strip width) x (average of first and last y-values, plus the sum of all **y** values between the second and second-last value)

Example #1

using a strip width of $\frac{\pi}{10}$, evaluate $\int_0^{\frac{2\pi}{5}} \tan x dx$

(answer to 2 d.p.)

$y_1 = \tan(0) = \quad 0$

$y_2 = \tan\left(\frac{\pi}{10}\right) = \quad 0.3249$

$y_3 = \tan\left(\frac{2\pi}{10}\right) = \quad 0.7265$

$y_4 = \tan\left(\frac{3\pi}{10}\right) = \quad \underline{1.3760}$

2.4274

$y_5 = \tan\left(\frac{4\pi}{10}\right) = \quad \underline{3.078}$

3.078

$$\int_0^{\frac{2\pi}{5}} \tan x dx = \frac{\pi}{10}\left[\frac{3.078}{2} + 2.4274\right]$$

$$= \frac{\pi}{10}\left[1.539 + 2.4274\right]$$

$$= 0.3142 \times 3.966 = 4.594$$

$$\underline{\int_0^{\frac{2\pi}{5}} \tan x dx \approx 4.59 \quad (2 \text{ d.p.})}$$

Example #2

evaluate $\int_3^8 x^2 dx$ using strips of width '1' unit.

x		y	y
3	y_1	9	
4	y_2		16
5	y_3		25
6	y_4		36
7	y_5		49
8	y_6	64	
		73	126

$$\int_3^8 x^2 dx = 1\left[\frac{73}{2} + 126\right]$$
$$= 36.5 + 126$$
$$= \underline{162.5}$$

Integrating Differential Equations

Introduction

All equations with derivatives of a variable w.r.t. another are called 'differential equations'.
A first order differential equation contains a first derivative eg dy/dx.

It might not be appreciated, but ALL integrals are derived from original 'first-order' differential equations.

$$\frac{dy}{dx} = f(x)$$
$$dy = f(x)dx$$
$$y = \int f(x)dx$$

Example:

$$\frac{dy}{dx} - x^2 + 3x - 2 = 0$$
$$\frac{dy}{dx} = x^2 - 3x + 2$$
$$dy = (x^2 - 3x + 2)dx$$
$$y = \int (x^2 - 3x + 2)dx$$
$$y = \frac{x^3}{3} - \frac{3x^2}{2} + 2x + C$$

('C' constant of integration)

First Order with 'variables separable'

Solution is by collecting all the 'y' terms on one side, all the 'x' terms on the other and integrating each expression independently.

$$\frac{dy}{dx} = f(x)g(y)$$

rearranging

$$\frac{1}{g(y)}\frac{dy}{dx} = f(x)$$

integrating both sides

$$\int \frac{1}{g(y)}\frac{dy}{dx}dx = \int f(x)dx$$

$$\Rightarrow \quad \int \frac{1}{g(y)}dy = \int f(x)$$

Example #1

$$\frac{dy}{dx} = \frac{4x^2+1}{3y}$$

rearranging

$$3y\frac{dy}{dx} = 4x^2+1$$

integrating both sides

$$\int 3y\frac{dy}{dx}dx = \int (4x^2+1)dx$$

$$\int 3y\,dy = \int (4x^2+1)dx$$

$$\frac{3y^2}{2} = \frac{4x^3}{3} + x + C_1$$

multiplying by 6

$$9y^2 = 8x^3 + 6x + C_2$$

dividing by 9

$$y^2 = \frac{8}{9}x^3 + \frac{2x}{3} + C_3$$

Note how the constant of integration C changes its value.

Example #2

$$\frac{dy}{dx} = 3xy^2$$

rearranging

$$\frac{1}{y^2}\frac{dy}{dx} = 3x$$

integrating both sides

$$\int \frac{1}{y^2}\frac{dy}{dx}dx = \int 3x dx$$

$$\int \frac{1}{y^2}dy = \int 3x dx$$

$$\int y^{-2}dy = \int 3x dx$$

$$-\frac{y^{-1}}{1} = \frac{3x^2}{2} + C_1$$

$$-\frac{1}{y} = \frac{3x^2}{2} + C_1$$

$$-2 = 3x^2 y + C_2$$

$$y = -\frac{2}{3x^2} + C_3$$

First Order 'linear' differential equations

By definition 'linear' differential equation have the form:

$$f(x)\frac{dy}{dx} + yg(x) = h(x)$$

Dividing by f(x) to make the coefficient of dy/dx equal to '1', the equation becomes:

$$\frac{dy}{dx} + Py = Q$$

(where P and Q are functions of x, and only x)

The key to solving these types of problem is to choose a multiplying factor(sometimes called an 'integrating factor') to make the LHS of the equation appear like a result from the Product Rule.

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Example

$$x^2\frac{dy}{dx}+3xy=2+x$$

$$\frac{dy}{dx}+\frac{3xy}{x^2}=\frac{2}{x^2}+\frac{x}{x^2}$$

$$\frac{dy}{dx}+\left(\frac{3}{x}\right)y=\frac{2}{x^2}+\frac{1}{x} \qquad \text{(i}$$

remember

$$\frac{dy}{dx}+Py=Q$$

$$\therefore\ P=\frac{3}{x},\quad Q=\frac{2}{x^2}+\frac{1}{x}$$

multiplying (i by x^3 to make the LHS like a Product Rule result

$$x^3\frac{dy}{dx}+3x^2y=2x+x^2$$

but $$\frac{d(x^3y)}{dx}=x^3\frac{dy}{dx}+3x^2y$$

$$\therefore\ \frac{d(x^3y)}{dx}=2x+x^2$$

integrating w.r.t. x

$$\int\frac{d(x^3y)}{dx}=\int(2x+x^2)dx$$

$$x^3y=\frac{2x^2}{2}+\frac{x^3}{3}+C$$

$$x^3y=x^2+\frac{x^3}{3}+C$$

Notes