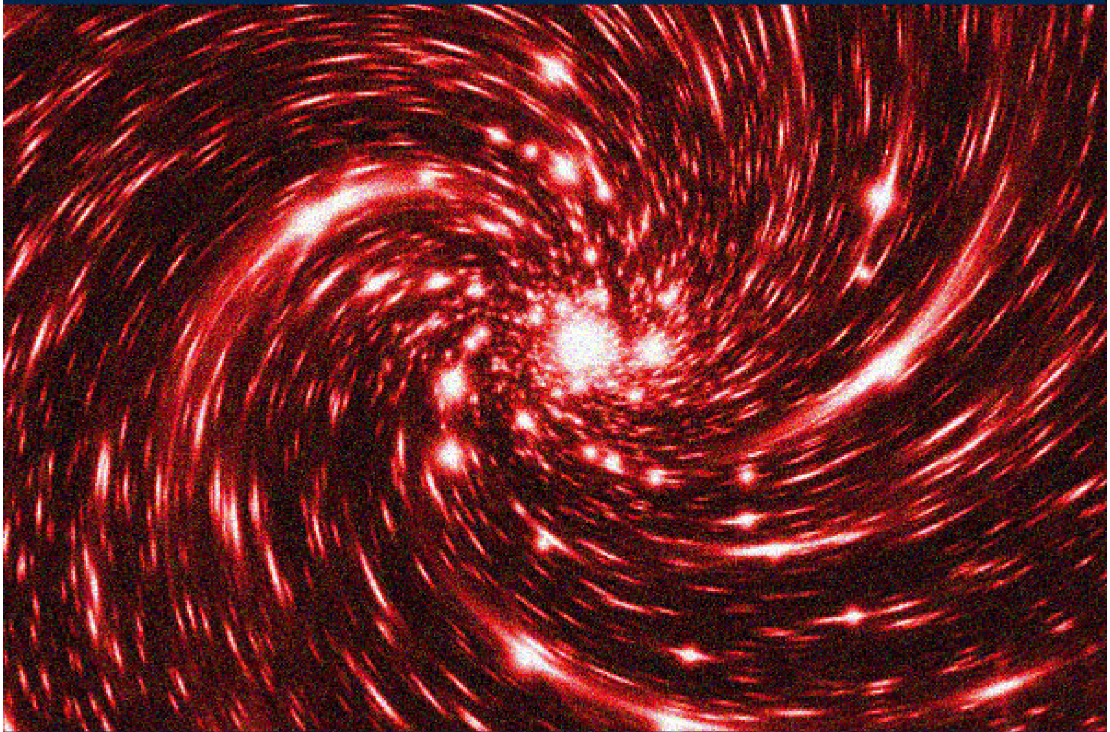
A-LEVEL MATHS TUTOR

Pure Maths



PART FOUR

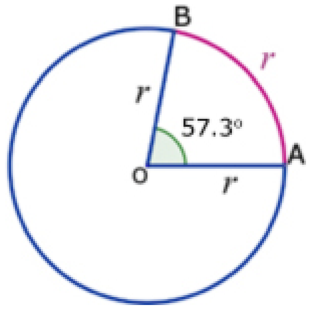
TRIGONOMETRY

www.a-levelmathstutor.com

Contents

Radians

What is a 'radian'?

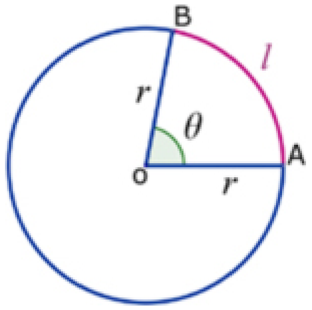


A radian is the angle subtended at the centre of a circle by an arc the same length as the radius of the circle.

Units

1^C (meaning 1 radian)= 57.296 deg.

Arc length



The arc length is proportional to its subtended angle.

Hence, if θ(theta) is in **degrees** and 'l' is the arc length:

$$\frac{l}{\theta} = \frac{2\pi r}{360}$$

$$l = \frac{\theta}{360} \times 2\pi r$$

An angle can be expressed in **radians** by dividing the arc length by the radius.

Therefore θ in radians is given by:

$$\theta = \frac{l}{r}$$

$$\therefore \text{ arc length } \quad \underline{l = r\theta}$$

Therefore for a circle(a 360 deg. angle), where the arc length is '2πr' and the radius is 'r' , the number of radians is 2πr/r , i.e. 2π .

Sector area

The area of a sector is proportional to the angle its arc subtends at the centre.

If a sector contains an angle of θ° then its area is given by:

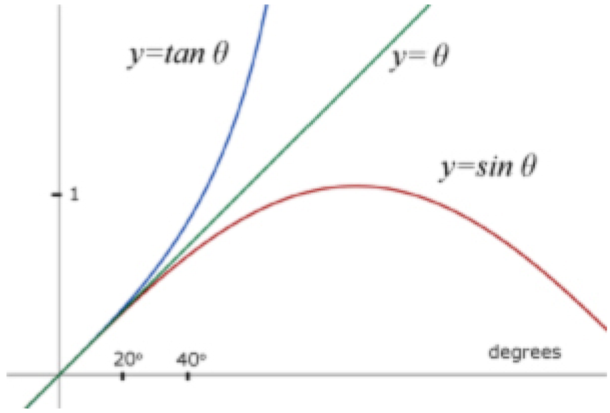
$$\text{area of sector } = \frac{\theta}{360} \times \pi r^2$$

However, if θ is in radians, remembering there are 2π radians in a circle:

$$\begin{aligned}\text{area of sector } &= \frac{\theta}{2\pi} \times \pi r^2 \\ &= \frac{\theta}{2} r^2\end{aligned}$$

$$\underline{\text{area of sector } = \frac{1}{2} r^2 \theta}$$

Small angles



For small angles(<10 deg.) there is a convergence between the value of the angle in radians with the value of its sine & tangent.

This approximate sine value may be expressed as:

$$\sin\theta \simeq \theta$$

The approximate cosine value is obtained thus:

$$\cos 2\theta = 1 - 2\sin^2\theta$$

$$\therefore \quad \cos\theta = 1 - 2\sin^2(\tfrac{1}{2}\theta) \qquad \text{(i}$$

if θ is small, $\sin\tfrac{1}{2}\theta \simeq \tfrac{1}{2}\theta$

hence equation (i becomes

$$\cos\theta = 1 - 2(\tfrac{1}{2}\theta)^2$$

$$\therefore \quad \underline{\cos\theta = 1 - \tfrac{1}{2}\theta^2}$$

Sine, Cosine & Tangent

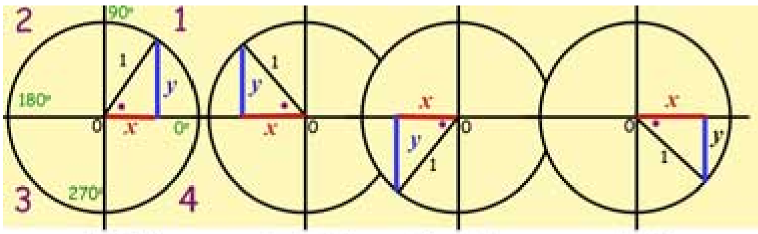
The General Angle

Consider a radius of length '1' rotating anti-clockwise about the origin. The coordinates of any point on the circle give the values of the adjacent and opposite sides of a right angled triangle, with the radius the hypotenuse.

The General Angle (theta) is the included angle between the radius and the x-coordinate of the point.

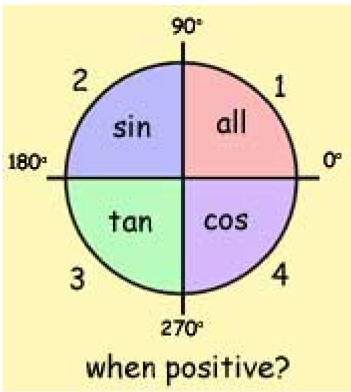
As the radius rotates the x and y values change. Hence the values of sine, cosine and tangent also change.

$$\sin\theta = \frac{y}{1} \quad \cos\theta = \frac{x}{1} \quad \tan\theta = \frac{y}{x}$$



	$x+$ $y+$	$x-$ $y+$	$x-$ $y-$	$x+$ $y-$
sin	+	+	-	-
cos	+	-	-	+
tan	+	-	+	-

The result is summarized in the diagram below.



Example #1

solve the equation $\sin\theta = -\frac{\sqrt{3}}{2}$ for $0 \le \theta \le 360$

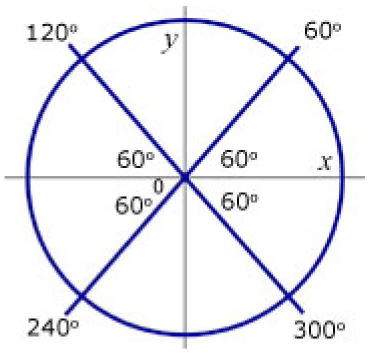
$\sin^{-1}(\frac{\sqrt{3}}{2}) = 60^\circ$

(meaning the angle whose sin is $\frac{\sqrt{3}}{2}$, is 60°)

the diagram shows the angles available with a

a sine value of $\pm\frac{\sqrt{3}}{2}$

however, the sine value is only negative in the 3rd and 4th quadrants



therefore the roots of the equation for $0 \le \theta \le 360$

are: $\underline{\theta = 240^\circ \quad \text{and} \quad \theta = 300^\circ}$

Example #2

solve the equation $2\cos 2\theta = \sqrt{3}$ for $-180^\circ < \theta < 180^\circ$

rewriting $2\cos 2\theta = \sqrt{3}$,

$$\cos 2\theta = \frac{\sqrt{3}}{2} = 0.8660$$

$$\cos^{-1} 2\theta = 60^\circ$$

$$\therefore \quad \theta = 30^\circ$$

N.B. if $-180^\circ < \theta < 180^\circ$ then $-360^\circ < 2\theta < 360^\circ$

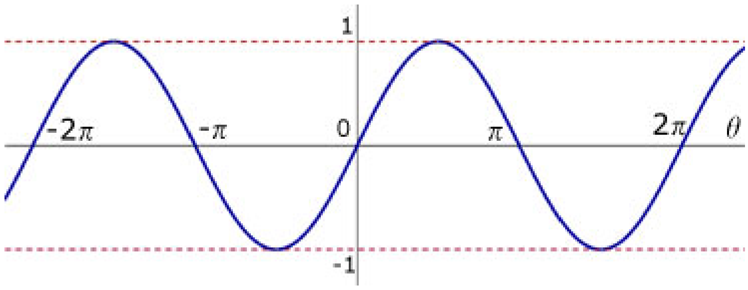
the cosine function is only + ve. in quadrants 1 & 4

$\therefore$ for $0 < 2\theta < 360^\circ$ $2\theta = 60^\circ, 300^\circ, -60^\circ, -300^\circ$

$\Rightarrow$ for $0 < \theta < 180^\circ$ $\theta = 30^\circ, 150^\circ, -30^\circ, -150^\circ$

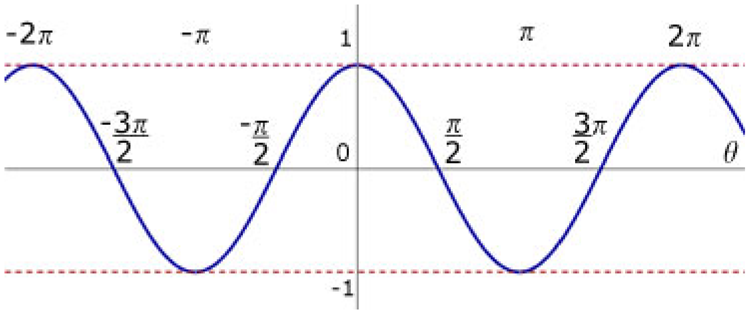
(remember + is an anticlockwise measure of angles
and - a clockwise measure)

Sine



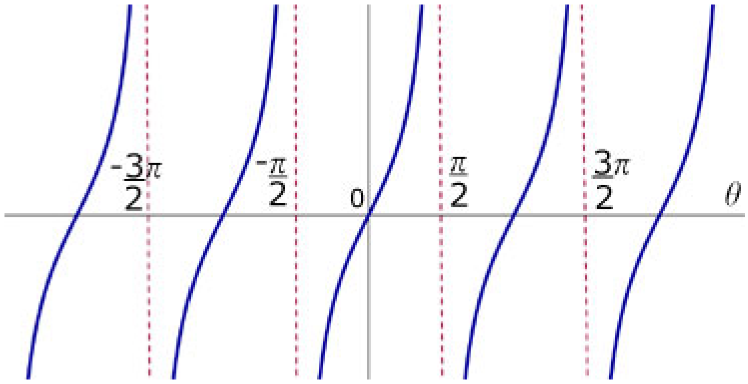
- the sine graph starts at **zero**
- it repeats itself every **360** degrees(or 2 pi)
- y is never more than **1** or less than **-1** (displacement from the x-axis is called the **amplitude**)
- a sin graph 'leads' a cos graph by 90 degrees

Cosine



- the cosine graph starts at **one**
- it repeats itself every **360** degrees(or 2 pi)
- y is never more than **1** or less than **-1** (displacement from the x-axis is called the **amplitude**)
- a cos graph 'lags' a sin graph by 90 degrees(pi/2) - this is termed a **phase shift**

Tangent



- the tangent graph starts at **zero**
- it repeats itself every **180** degrees
- y can vary between numbers approaching infinity and minus infinity

Further comparison

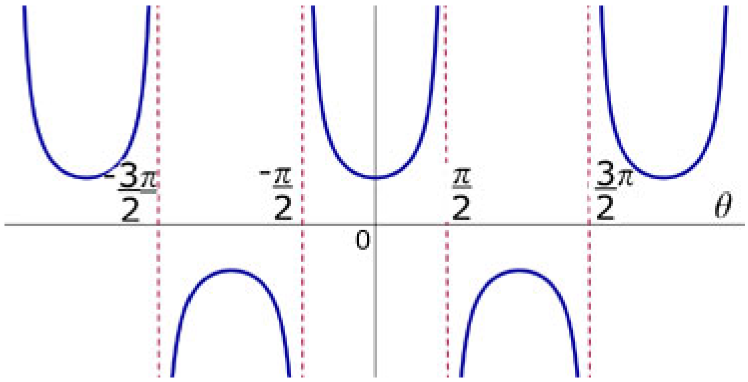
- only the cosine function is symmetrical about the y-axis
- all the functions are **cyclic** - the distance along the horizontal axis repeated is called **the period**

Cosecant, Secant & Cotangent

Introduction

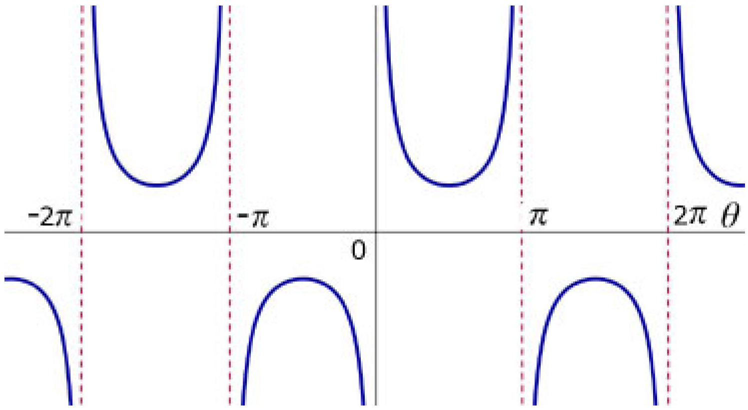
$$\text{cosec}\,\theta = \frac{1}{\sin\theta}$$
$$\sec\theta = \frac{1}{\cos\theta}$$
$$\cot\theta = \frac{1}{\tan\theta}$$

Secant (sec)



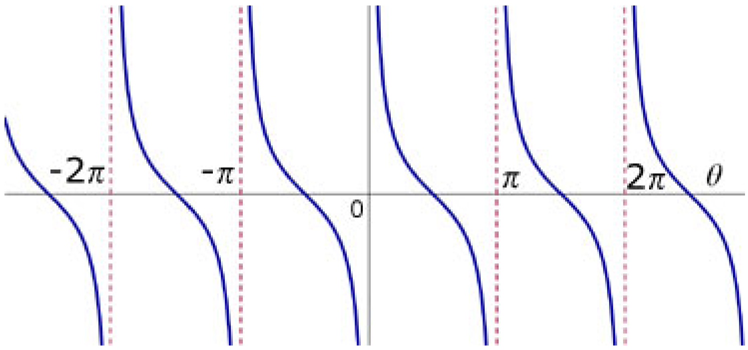
- the secant graph is symmetrical about the y-axis
- it repeats itself every **360** degrees- period 2Π
- y can vary between numbers approaching infinity and minus infinity
- asymptotes start at + and - 90 degrees(Π/2) and at continue at intervals of 180 degrees(Π) after that
- the asymptotes also correspond to the x-intercepts for cos(x)
- the minima along the x-axis correspond to the maxima of the cosine function(and vice versa)

Cosecant (cosec)



- the cosecant graph is **NOT** symmetrical about the y-axis
- it repeats itself every **360** degrees - period 2Π
- y can vary between numbers approaching infinity and minus infinity
- asymptotes start at zero and + and - 180 degrees(Π) and at intervals of 180 degrees(Π) after that
- the asymptotes also correspond to the x-intercepts for sin(x)
- the minima along the x-axis correspond to the maxima of the sine function(and vice versa)

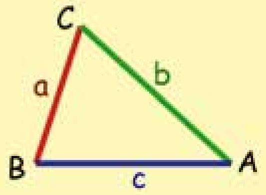
Cotangent (cot)



- the cotangent graph is **NOT** symmetrical about the y-axis
- it repeats itself every **180** degrees - period Π
- y can vary between numbers approaching infinity and minus infinity
- asymptotes start at zero and + and - 180 degrees(Π) and at intervals of 180 degrees(Π) after that
- the x-asymptotes correspond to the *x*-intercepts of the function y = tan(*x*)
- y = tan(*x*) and y = cot(*x*) face in opposit directions - (**tan** has a **positive gradient** while **cot** is **negative**)

Sine Rule, Cosine Rule

The Sine Rule



$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

- Use either the right, or left hand equation.
- You are given 3 quantities and required to work out the 4 th.

Manipulating the ratio - Take two ratios, cross multiply and rearrange to put the required quantity as the subject of the equation.

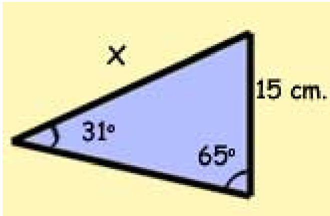
$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$a \sin B = b \sin A$$

$$a = \frac{b \sin A}{\sin B} \qquad b = \frac{a \sin B}{\sin A}$$

$$\sin B = \frac{b \sin A}{a} \qquad \sin A = \frac{a \sin B}{b}$$

Example #1



$$\frac{15}{\sin 31^\circ} = \frac{x}{\sin 65^\circ}$$

$$x\sin 31^\circ = 15\sin 65^\circ$$

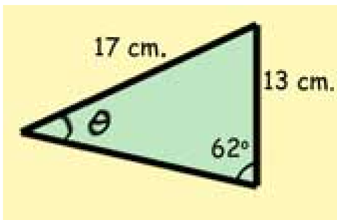
$$x = \frac{15\sin 65^\circ}{x\sin 31^\circ}$$

$$= \frac{15\times 0.9063}{0.5150}$$

$$= 26.3971$$

$\underline{x = 26.40\ \text{cm}}$ (2 d.p.)

Example #2



$$\frac{17}{\sin 62^\circ} = \frac{13}{\sin\theta}$$

$$17\sin\theta = 13\sin 62^\circ$$

$$\sin\theta = \frac{13\sin 62^\circ}{17}$$

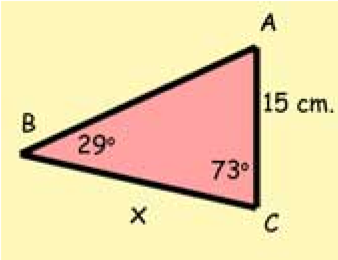
$$= \frac{13\times 0.8829}{17}$$

$$= 0.6752$$

$$\theta = 42.4697^\circ$$

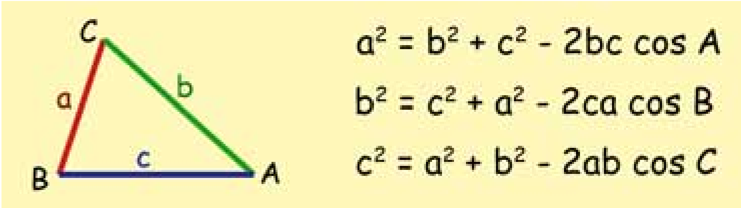
$\underline{\theta = 42.47^\circ}$ (2 d.p.)

Example #3



$$\text{angle A} = 180^\circ - 29^\circ - 73^\circ$$
$$= 78^\circ$$
$$\frac{x}{\sin 78^\circ} = \frac{15}{\sin 29^\circ}$$
$$x \sin 29^\circ = 15 \sin 78^\circ$$
$$x = \frac{15 \sin 78^\circ}{\sin 29^\circ}$$
$$= \frac{15 \times 0.9781}{0.4848}$$
$$= 30.2630$$
$$\underline{x = 30.26 \text{ cm}} \ (2 \text{ d.p.})$$

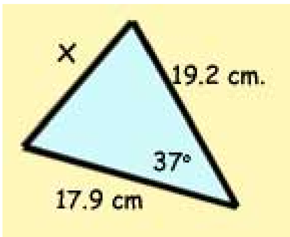
The Cosine Rule



There are two problem types:

- You are given 2 sides + an included angle and required to work out the remaining side
- You are given all the sides and required to work out the angle.

Example #1



$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$x^2 = (19.2)^2 + (17.90)^2 - \left[2 \times 19.2 \times 17.9 \times \cos 37^\circ\right]$$

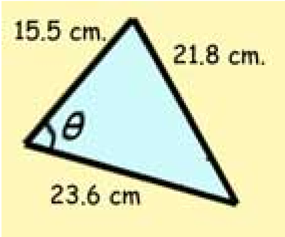
$$= (368.64) + (320.4) - (687.36) \times (0.7986)$$

$$= 140.1243$$

$$x = 11.8374$$

$$= \underline{11.84\text{ cm}}\ (2\text{ d.p.})$$

Example #2



$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos\theta = \frac{(23.6)^2 + (15.5)^2 - (21.8)^2}{2(23.6)(15.5)}$$

$$= \frac{(556.96) + (240.25) - (475.24)}{731.60}$$

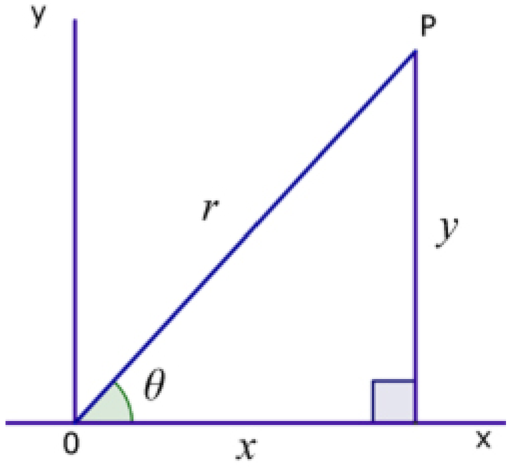
$$= \frac{321.97}{731.60} = 0.4401$$

$$\theta = 63.8904^\circ$$

$$= \underline{63.89^\circ}\ (2.\text{d.p.})$$

Pythagorean Identities

identity #1 $\cos^2\theta + \sin^2\theta \equiv 1$



by Pythagoras' Theorem:

$$x^2 + y^2 = r^2 \qquad \text{(i}$$

dividing (i by r^2

$$\Rightarrow \qquad \frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$$

but $\cos\theta = \dfrac{x}{r}$ and $\sin\theta = \dfrac{y}{r}$

$\therefore$ $\underline{\cos^2\theta + \sin^2\theta \equiv 1}$

Example

solve the equation $2\sin^2\theta = 1 - \cos\theta$

for θ where $0 \le \theta < 360°$

$2\sin^2\theta = 1 - \cos\theta$ (i

using $\cos^2\theta + \sin^2\theta = 1$

$\Rightarrow \quad \sin^2\theta = 1 - \cos^2\theta$

substituting for $\sin^2\theta$ into (i

$2(1 - \cos^2\theta) = 1 - \cos\theta$

$2 - 2\cos^2\theta = 1 - \cos\theta$

$0 = 1 - \cos\theta - 2 + 2\cos^2\theta$

$0 = 2\cos^2\theta - \cos\theta - 1$

$2\cos^2\theta - \cos\theta - 1 = 0$

$(2\cos\theta + 1)(\cos\theta - 1) = 0$

$\therefore \quad \cos\theta = -\tfrac{1}{2} \quad \text{or} \quad \cos\theta = 1$

$\Rightarrow \quad \underline{\theta = 60°,\ 120°,\ 240° \quad \text{or} \quad \theta = 0°}$

identity #2 $1+\tan^2\theta \equiv \sec^2\theta$

dividing $\cos^2\theta + \sin^2\theta \equiv 1$ by $\cos^2\theta$

$$1+\frac{\sin^2\theta}{\cos^2\theta} \equiv \frac{1}{\cos^2\theta}$$

but $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ and $\sec\theta = \dfrac{1}{\cos\theta}$

$\therefore$ $\underline{1+\tan^2\theta \equiv \sec^2\theta}$

Example

solve the equation $3\tan\theta = \sec^2\theta + 1$

for $0 \le \theta \le 360^\circ$

rearranging

$$0 = \sec^2\theta - 3\tan\theta + 1$$

$$\sec^2\theta - 3\tan\theta + 1 = 0 \qquad \text{(i}$$

using the identity

$$1+\tan^2\theta \equiv \sec^2\theta$$

substituting for $\sec^2\theta$ into (i

$$(1+\tan^2\theta) - 3\tan\theta + 1 = 0$$

$$\tan^2\theta - 3\tan\theta + 2 = 0$$

$$(\tan\theta - 1)(\tan\theta - 2) = 0$$

$\tan\theta = 1$ or $\tan\theta = 2$

$\therefore$ $\theta = 45^\circ, 225^\circ$ or $\theta = 63.4^\circ, 243.4$

identity #3 $\cot^2 + 1 \equiv \mathrm{cosec}^2\,\theta$

dividing $\cos^2\theta + \sin^2\theta \equiv 1$ by $\sin^2\theta$

$$\frac{\cos^2\theta}{\sin^2\theta} \equiv \frac{1}{\sin^2\theta}$$

but $\mathrm{cotan}\,\theta = \dfrac{\cos\theta}{\sin\theta}$ and $\mathrm{cosec}\,\theta = \dfrac{1}{\sin\theta}$

$\therefore$ $\cot^2\theta + 1 \equiv \mathrm{cosec}^2\theta$

Example

simplify $\left[x^2\left(\mathrm{cosec}^2\theta - 1\right)\right]^{-1}$

expanding

$$\left[x^2\left(\mathrm{cosec}^2\theta - 1\right)\right]^{-1} \equiv \frac{1}{x^2\left(\mathrm{cosec}^2\theta - 1\right)}$$

$$\equiv \frac{1}{x^2\mathrm{cosec}^2\theta - x^2} \qquad \text{(i}$$

using the identity

$$\cot^2\theta + 1 = \mathrm{cosec}^2\theta$$

and substituting for $\mathrm{cosec}^2\theta$ into (i

$$\left[x^2\left(\mathrm{cosec}^2\theta - 1\right)\right]^{-1} \equiv \frac{1}{x^2(\cot^2\theta + 1) - x^2}$$

$$\equiv \frac{1}{x^2\cot^2\theta + x^2 - x^2}$$

$$\equiv \frac{1}{x^2\cot^2\theta}$$

but $\tan\theta = \dfrac{1}{\cot\theta}$

$\therefore$ $\left[x^2\left(\mathrm{cosec}^2\theta - 1\right)\right]^{-1} \equiv \dfrac{\tan^2\theta}{x^2}$

Compound Angle Formulae

The Six Compound Angle Identities

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$
$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$
$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A+B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

Double Angle Formulae

making $A = B$,

$$\sin 2A = 2(\sin A \cos A)$$
$$\cos 2A = \cos^2 A - \sin^2 A$$

$$\cos 2A = \cos^2 A - \sin^2 A \qquad (i$$

but $\quad \sin^2 A + \cos^2 A = 1$

$\Rightarrow \quad \cos^2 A = 1 - \sin^2 A$

$\Rightarrow \quad \sin^2 A = 1 - \cos^2 A$

substituting for $\cos^2 A$ in (i

$$\cos 2A = 1 - \sin^2 A - \sin^2 A$$
$$\underline{\cos 2A = 1 - 2\sin^2 A}$$

substituting for $\sin^2 A$ in (i

$$\cos 2A = \cos^2 A - (1 - \cos^2 A)$$
$$\cos 2A = \cos^2 A - 1 + \cos^2 A)$$
$$\underline{\cos 2A = 2\cos^2 A - 1}$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Factor Formulae

$$\cos(A+B)+\cos(A-B)=2\cos A\cos B$$
$$\cos(A+B)-\cos(A-B)=-2\sin A\sin B$$
$$\sin(A+B)+\sin(A-B)=2\sin A\cos B$$
$$\sin(A+B)-\sin(A-B)=2\cos A\sin B$$

let $P=A+B$ and $Q=A-B$

adding $\quad P+Q=2A$

$\therefore \quad \underline{A=\dfrac{P+Q}{2}}$

subtracting $\quad P-Q=2B$

$\therefore \quad \underline{B=\dfrac{P-Q}{2}}$

substituting for A, B above

$$\cos P+\cos Q=2\cos\frac{P+Q}{2}\cos\frac{P-Q}{2}$$
$$\cos P-\cos Q=-2\sin\frac{P+Q}{2}\sin\frac{P-Q}{2}$$
$$\sin P+\sin Q=2\sin\frac{P+Q}{2}\cos\frac{P-Q}{2}$$
$$\sin P-\sin Q=2\cos\frac{P+Q}{2}\sin\frac{P-Q}{2}$$

rcos() form

Adding a sine and a cosine will generate a cosine curve. This will have a larger amplitude than the original and is out of phase with it.

Writing the expression as **rcos(θ - α)**,

'**r**' amplitude
'**α**' no. degrees phase difference(to the right)

Thus expressions of the form:

$$a\cos\theta + b\sin\theta$$

can be rewritten as

$$r\cos(\theta + \alpha)$$

Finding 'r' and the phase angle 'α'

$$r\cos(\theta+\alpha) = r\ \cos\theta\ \cos\alpha - r\ \sin\theta\ \sin\alpha$$
$$= r\ \cos\alpha\ \cos\theta - r\ \sin\alpha\ \sin\theta$$

let

$a = r\cos\alpha$ (i

$b = -r\sin\alpha$ (ii

so that

$(r\ \cos\alpha)\ \cos\theta + (-r\ \sin\alpha)\ \sin\theta = a\cos\theta\ +\ b\sin\theta$

α is found by dividing (ii by (i

$$\frac{b}{a} = \frac{-r\sin\alpha}{r\cos\alpha}$$

$$\Rightarrow \qquad \underline{\tan\alpha = -\frac{b}{a}}$$

r is found by squaring and then adding (i & (ii

$a^2 + b^2 = r^2\cos^2\alpha + r^2\sin^2\alpha$

$a^2 + b^2 = r^2(\cos^2\alpha + \sin^2\alpha)$

but $\cos^2\alpha + \sin^2\alpha = 1$

$\therefore \qquad a^2 + b^2 = r^2$

$$\underline{r = \sqrt{a^2 + b^2}}$$

n.b. $a\cos\theta\ +\ b\sin\theta$ can also be written using the sine expression $r\sin(\theta+\alpha)$

Notes