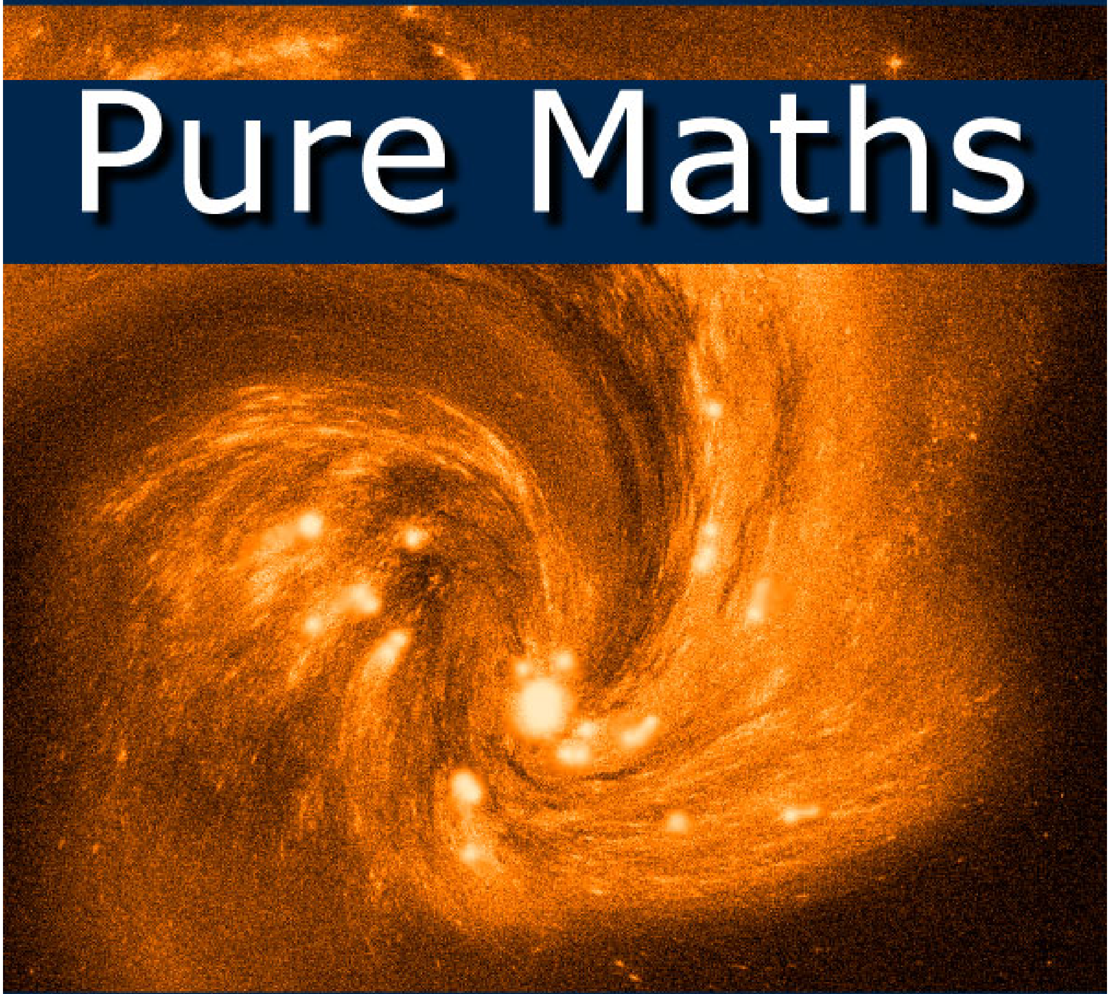
A-LEVEL MATHS TUTOR

Pure Maths



PART SEVEN

VECTORS

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Contents

General Properties of Vectors

Notation

A non-zero vector is has the magnitude of a positive real number and a direction in space.

A vector may be represented by two letters describing a line.The order of the letters indicates the direction and the length of the line its magnitude.

$$\text{vector } \overrightarrow{AB} \qquad \text{magnitude } \left|\overrightarrow{AB}\right|$$

An alternative to this notation is to use a single bold letter, for example **C**. Then the magnitude is |**C**| or C.

The Unit Vector

A unit vector eg **a** , has a magnitude of one |**a**|=1 and can point in any direction.

Sometimes a unit vector is written with an accent over it **â**.

Different unit vectors point in different directions.

Hence, if **F** is an ordinary vector,

$$\frac{F}{|F|} = \hat{F}$$

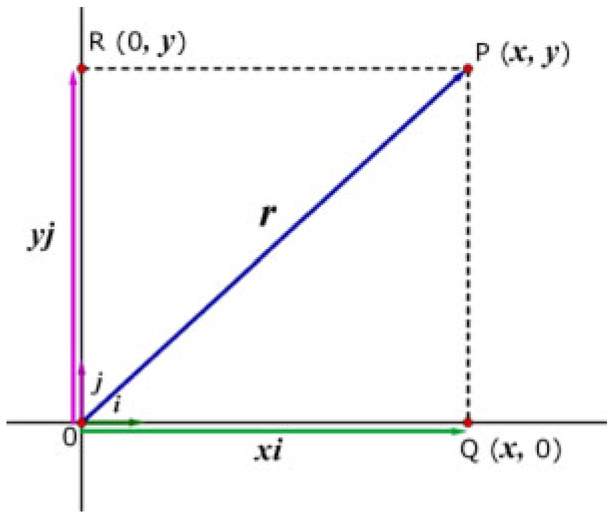
a unit vector in the direction of **F** (^ circumflex accent)

<u>2D representation</u>

P is a point in the x-y plane with coodinates (x,y). $\boldsymbol{i}$ is the unit vector along the x-axis and $\boldsymbol{j}$ is the unit vector along the y-axis.

With Q at (x,0) and R at (0, y):

$$\therefore \quad \overrightarrow{OP} = \overrightarrow{OQ} + \overrightarrow{QP}$$
$$= \overrightarrow{OQ} + \overrightarrow{OR}$$
$$\Rightarrow \quad \mathbf{r} = x\mathbf{i} + y\mathbf{j}$$

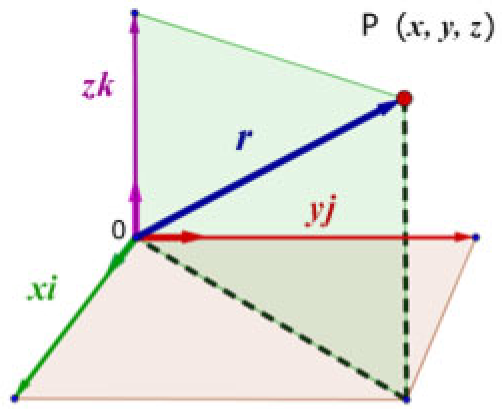


3D representation

P is a point in x-y-z space with coodinates (x, y, z). $\boldsymbol{i}$ is the unit vector along the x-axis, $\boldsymbol{j}$ is the unit vector along the y-axis and $\boldsymbol{z}$ is the unit vector along the z-axis.

$$\overrightarrow{OP} = r = xi + yj + zk$$

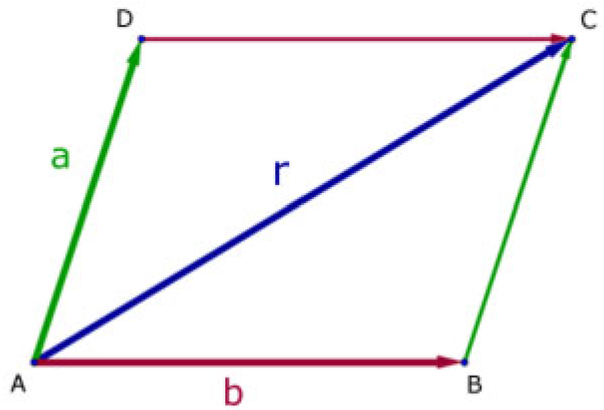
$$\text{length OP} = |\overrightarrow{OP}|$$
$$= \sqrt{(x^2 + y^2 + z^2)}$$



Addition(Sum) of vectors

This is also called the Parallelogram or Triangle Law.

If two vectors(**a** & **b**) are represented in magnitude and direction by the adjacent sides of a **parallelogram** from a point, then their resultant(**r**) is represented in magnitude and direction by the diagonal of the parallelogram passing through the point.

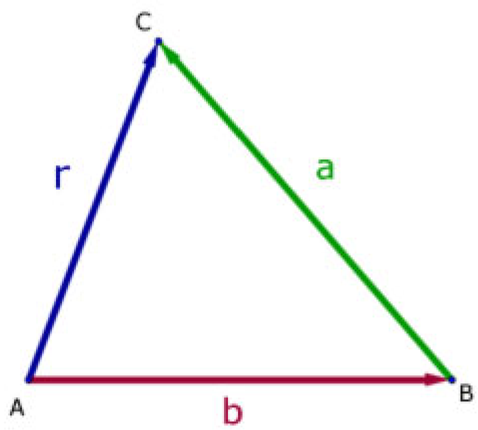


$$\overrightarrow{AB} + \overrightarrow{AD} = \overrightarrow{AC}$$

$$a = \overrightarrow{AB} \quad b = \overrightarrow{AD}$$

$$a + b = r$$

If two vectors are represented in magnitude and direction by the adjacent sides of a **triangle**, taken in order, then their resultant is represented in magnitude but opposite in direction by the third side.



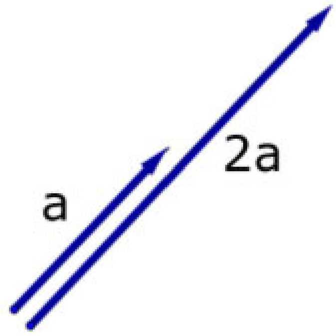
$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

$$b = \overrightarrow{AB} \quad a = \overrightarrow{BC}$$

$$b + a = r$$

Scalar Multiplication

Multiplying a vector by a scalar quantity changes its magnitude but not its direction.



Vector Equations

Component rules

Consider two vectors:

$$\mathbf{a} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k} \quad \text{and} \quad \mathbf{b} = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$$

in three dimensional space.

a = **b** implies that $x_1 = x_2$ $y_1 = y_2$ $z_1 = z_2$

$\mathbf{a} + \mathbf{b} = (x_1 + x_2)\mathbf{i} + (y_1 + y_2)\mathbf{j} + (z_1 + z_2)\mathbf{k}$

$\mathbf{a} - \mathbf{b} = (x_1 - x_2)\mathbf{i} + (y_1 - y_2)\mathbf{j} + (z_1 - z_2)\mathbf{k}$

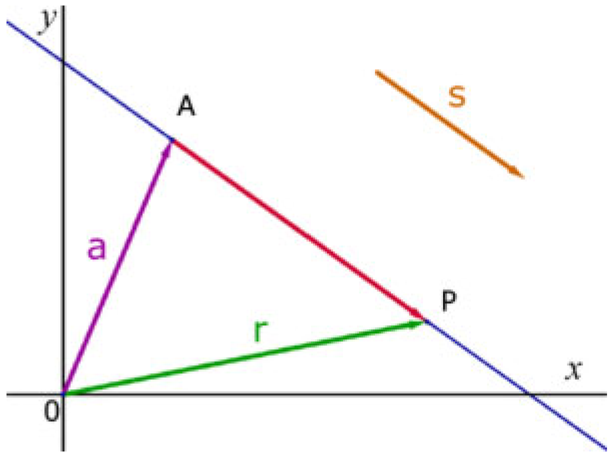
$m\mathbf{a} = mx_1\mathbf{i} + m\,y_1\mathbf{j} + mz_1\mathbf{k}$ where m is a scalar quantity

$$|\mathrm{a}| = \sqrt{\left(x_1^2 + y_1^2 + z_1^2\right)} \qquad |\mathrm{b}| = \sqrt{\left(x_2^2 + y_2^2 + z_2^2\right)}$$

in 3D space, if point A has position vector **a** and point B has position vector **b** then the distance AB is given by:

$$\mathrm{AB} = \left|\overrightarrow{\mathrm{AB}}\right| = \mathrm{b} - \mathrm{a}$$
$$= (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$$

Equation of a straight line - single point & parallel vector given



A(x_1 y_1 z_1) is a fixed point on the line

a is the position vector for point A **a** = x_1**i** + y_1 **j**+ z_1**k**

s is a vector parallel to the line **s** = *l* **i** + *m* **j**+ *n* **k**
(*l m n* are called the direction ratios of the line)

r is the position vector for an arbitrary point P(x, y, z) on the line.

$\overrightarrow{AP}$ is parallel to s

$\Rightarrow \overrightarrow{AP} = \sigma s$ (σ is a scalar variable)

$\Rightarrow$ $\underline{r = a + \sigma s}$ (vector equation of the line)

substituting for a and s

$r = x_1 i + y_1 j + z_1 k + \sigma(li + mj + nk)$

explanding and rearranging

$r = (x_1 + \sigma l)i + (y_1 + \sigma m)j + (z_1 + \sigma n)k$

$r = xi + yj + zk$

comparing the coefficients of i j k

$x = x_1 + \sigma l \qquad y = y_1 + \sigma m \qquad z = z_1 + \sigma n$

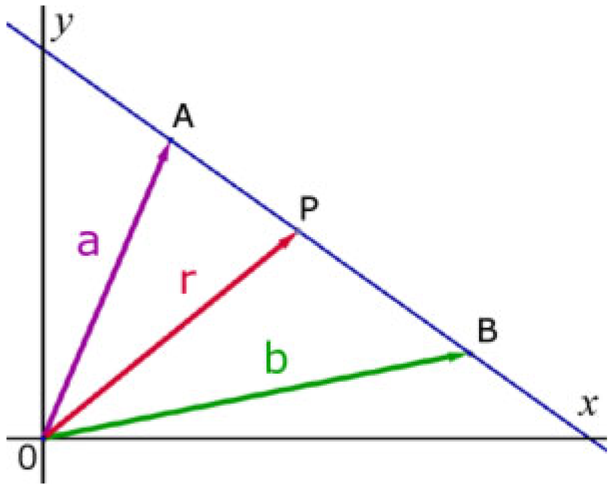
rearranging to make σ the subject

$\sigma = \frac{x - x_1}{l}, \qquad \sigma = \frac{y - y_1}{m}, \qquad \sigma = \frac{z - z_1}{n}$

hence the Cartesian equation of the line:

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} \quad (= \sigma)$$

Equation of a straight line - two points given



A(x_1 y_1 z_1) is a fixed point on the line

a is the position vector for point A **a** = x_1**i** + y_1 **j**+ z_1**k**

B(x_2 y_2 z_2) is a fixed point on the line

b is the position vector for point B **b** = x_2**i** + y_2 **j**+ z_2**k**

r is the position vector for an arbitrary point P(x, y, z) on the line.

let $\overrightarrow{AB}$ be a vector from A to B

$\Rightarrow \quad \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

since $\quad b = \overrightarrow{OB} \quad$ and $\quad a = \overrightarrow{OA}$

$\therefore \quad \overrightarrow{AB} = b - a \qquad\qquad$ (i

$\overrightarrow{AP} = \theta\overrightarrow{AB} \qquad\qquad$ (θ is a scalar variable)

substituting for $\overrightarrow{AB}$ from (i

$\overrightarrow{AP} = \theta(b - a)$

$\overrightarrow{OP} = r$

$r = \overrightarrow{OA} + \overrightarrow{AP}$

$\underline{r = a + \theta(b - a)} \qquad$ (vector equation of the line)

replacing the vectors by their components

$$xi + yj + zk = x_1i + y_1j + z_1k + \theta(x_2i + y_2j + z_2k - x_1i + y_1j + z_1k)$$

expanding

$$xi + yj + zk = x_1i + y_1j + z_1k + \theta x_2i + \theta y_2j + \theta z_2k - \theta x_1i + \theta y_1j + \theta z_1k$$

factorising

$$xi + yj + zk = [x_1 + \theta(x_2 - x_1)]i + [y_1 + \theta(y_2 - y_1)]j + [z_1 + \theta(z_2 - z_1)]k$$

comparing the coefficients of $\quad i \;\; j \;\; k$

$x = x_1 + \theta(x_2 - x_1)$

$y = y_1 + \theta(y_2 - y_1)$

$z = z_1 + \theta(z_2 - z_1)$

making θ the subject

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} \quad (= \theta)$$

(the Cartesian equations)

The Scalar Product

Introduction

The Scalar(or Dot Product), of two vectors **a** and **b** is written

$$\mathbf{a}.\mathbf{b}$$

If the two vectors are inclined to each other by an angle(say θ) then the product is written

$$\mathbf{a}.\mathbf{b} = |\mathbf{a}|.|\mathbf{b}|\cos\theta \quad \text{or} \quad \mathbf{a}.\mathbf{b} = ab\cos\theta$$

Even though the left hand side of the equation is written in terms of vectors, the answer is a scalar quantity.

Rules

$\mathbf{a}.\mathbf{b} = ab\cos\theta = \mathbf{b}.\mathbf{a}$

when **a** & **b** are parallel, $\theta = 0$, $\cos\theta = 1$, $\mathbf{a}.\mathbf{b} = ab$
(unit vectors $\mathbf{i}.\mathbf{i} = \mathbf{j}.\mathbf{j} = \mathbf{k}.\mathbf{k} = 1$)

when **a** & **b** are at 90°, $\theta = 90°$, $\cos\theta = 0$, $\mathbf{a}.\mathbf{b} = 0$
(unit vectors: $\mathbf{i}.\mathbf{j} = \mathbf{j}.\mathbf{i} = 0$ $\mathbf{j}.\mathbf{k} = \mathbf{k}.\mathbf{j} = 0$ $\mathbf{k}.\mathbf{i} = \mathbf{i}.\mathbf{k} = 0$)

if $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$

then

$$\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

$$|\mathbf{a}|^2 = \mathbf{a}.\mathbf{a} = a_1^2 + a_2^2 + a_3^2$$

$\mathbf{a}.(\mathbf{b} + \mathbf{c}) = \mathbf{a}.\mathbf{b} + \mathbf{a}.\mathbf{c}$ $\quad$ $\mathbf{a}.(\mathbf{b} - \mathbf{c}) = \mathbf{a}.\mathbf{b} - \mathbf{a}.\mathbf{c}$

$(\mathbf{a} + \mathbf{b}).\mathbf{c} = \mathbf{a}.\mathbf{c} + \mathbf{b}.\mathbf{c}$ $\quad$ $(\mathbf{a} - \mathbf{b}).\mathbf{c} = \mathbf{a}.\mathbf{c} - \mathbf{b}.\mathbf{c}$

$(\lambda\mathbf{a}).\mathbf{b} = \lambda(\mathbf{a}.\mathbf{b}) = \mathbf{a}.(\lambda\mathbf{b})$

$$\cos\theta = \frac{a.b}{|a|.|b|}$$

Example #1

Given that **a** = 3**i** - **j** + 2**k** and **b** = 2**i** + **j** - 2**k** ,
find **a**.**b** and the included angle between the vectors to 1 d.p.

$$\begin{aligned} a.b &= (3i - j + 2k).(2i + j - 2k) \\ &= 6 - 1 - 4 \\ &= 1 \end{aligned}$$

$$\underline{a.b = 1}$$

$$\begin{aligned} |a^2| &= a.a \\ &= (3i - j + 2k).(3i - j + 2k) \\ &= 9 + 1 + 4 \\ &= 14 \\ |a| &= \sqrt{14} \end{aligned}$$

$$\begin{aligned} |b^2| &= b.b \\ &= (2i + j - 2k).(2i + j - 2k) \\ &= 4 + 1 + 4 \\ &= 9 \\ |b| &= 3 \end{aligned}$$

with θ the included angle

$$\cos\theta = \frac{a.b}{|a|.|b|}$$

substituting for $a.b$, $|a|$, $|b|$

$$\cos\theta = \frac{1}{\sqrt{14} \times 3} = \frac{1}{11.225}$$

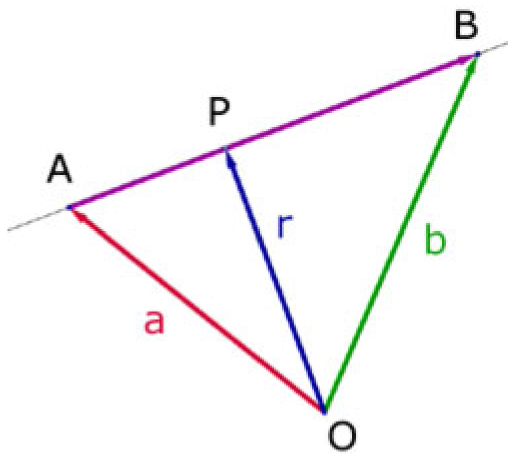
$$\Rightarrow \quad \theta = 84.89^\circ$$

<u>the angle between a and b is 84.9° (1 d.p.)</u>

Example #2

What is the vector equation describing the straight line passing through the points A(-8, 1, -2) and B(10, -1, 3)?

Find the coordinates of a point P on AB such that OP is perpendicular to AB(origin O), hence find the distance OP to 2d.p.



given points: A(-8, 1, -2) B(10, -1, 3)

$\Rightarrow \ \overrightarrow{AB} = (-8+10)i + (1-1)j + (-2+3)k$

$\quad = 2i \ + \ 0 \ + \ k$

$\quad = 2i \ + \ k$

if r is the position vector for point P

the vector equation of line AB is:

$\underline{r = a + \lambda(\overrightarrow{AB})}$

$r = a + \lambda(2i + k)$

$= -8i + j - 2k + \lambda(2i + k)$

$= -8i + j - 2k + 2\lambda i + \lambda k$

$= (2\lambda - 8)i + j + (\lambda - 2)k$

$\therefore$ coords. of any point P on AB are $(2\lambda - 8, 1, \lambda - 2)$

for $\overrightarrow{OP}$ to be perpendicular to $\overrightarrow{AB}$

$\Rightarrow \ \overrightarrow{OP} \, . \, \overrightarrow{AB} = 0$

$\Rightarrow \ [(2\lambda - 8)i + j + (\lambda - 2)k] \, . \, [(2i + k)] = 0$

$\Rightarrow \ 2(2\lambda - 8) + 0 + 1(\lambda - 2) = 0$

$\Rightarrow \ 4\lambda - 16 + \lambda - 2 = 0$

$\Rightarrow \ 5\lambda = 18$

$\Rightarrow \ \lambda = 3.6$

substituting into coords. for $P(2\lambda - 8, 1, \lambda - 2)$

$\Rightarrow \ P(2 \times 3.6 - 8, \ 1, \ 3.6 - 2)$

$\Rightarrow \ P(-0.8, \ 1, \ 1.6)$

$\therefore \ |\overrightarrow{OP}| = \sqrt{(-0.8)^2 + (1)^2 + (1.6)^2}$

$\quad = \sqrt{(0.64) + (1) + (2.56)}$

$\quad = 2.04939$

perp. dist. between O and AB is 2.05 (2d.p.)

Notes