Calculus: Maximum & Minima

Gradient change

Starting to the left of a maximum the gradient changes from '+ ' to ' - ' with increasing 'x'.

Starting to the left of a minimum, the gradient changes from ' - ' to '+ ' with increasing 'x'.

At the point of maximum or minimum the gradient is zero.
Example

Show that the curve \( y = x^2 \) has a minimum at \((0,0)\).

\[
\begin{align*}
    y &= x^2, \\
    \text{gradient, } \frac{dy}{dx} &= 2x \\
    \text{at the point } (0,0) &\Rightarrow x = 0 \\
    \text{gradient, } \frac{dy}{dx} &= 2(0) = 0 \\
\end{align*}
\]

\( \therefore \) there is either a maximum or minimum at \((0,0)\) taking a value of \( x \) less than 0, say \(-1\)

\[
\begin{align*}
    \text{gradient, } \frac{dy}{dx} &= 2(-1) = -2 \text{ (a negative gradient)} \\
\end{align*}
\]

taking a value of \( x \) more than 0, say \( +1 \)

\[
\begin{align*}
    \text{gradient, } \frac{dy}{dx} &= 2(1) = 2 \text{ (a positive gradient)} \\
\end{align*}
\]

The gradient changes from negative to positive with increasing \( x \).

\( \therefore \) the function has a minimum at \((0,0)\)
Locating the point of maximum or minimum.

The x-value at a maximum or minimum is found by differentiating the function and putting it equal to zero.

The y-value is then found by substituting the 'x' into the original equation.

Example

Find the coordinates of the greatest or least value of the function:

\[ y = x^2 + 3x + 2 \]

\[
\text{gradient, } \frac{dy}{dx} = 2x + 3
\]

max. or min. when gradient is zero

0 = 2x + 3

\[ x = -\frac{3}{2} = -1.5 \]

substituting this value of x into \( y = x^2 + 3x + 2 \)

\[ y = \left(-\frac{3}{2}\right)^2 + 3\left(-\frac{3}{2}\right) + 2 \]

\[ y = \left(\frac{9}{4}\right) - \left(\frac{9}{2}\right) + 2 \]

= 2.25 - 4.5 + 2 = -0.25

coords. of the maximum/minimum are \((-1.5, -0.25)\)
**Curve Sketching**

The power of 'x' gives a hint to the general shape of a curve.

Together with the point of maximum or minimum, where the curve crosses the axes at $y=0$ and $x=0$ gives further points.
Example

Sketch the curve \( y = x^2 + 3x + 2 \) from the example above, given that there is a minimum at \((-1.5, -0.25)\).

Factorising and putting \( y = 0 \) to find where the curve crosses the x-axis,

\[
(x+1)(x+2)=0
\]

\( x = -1 \) and \( x = -2 \)

so the curve crosses the x-axis at \((-1, 0)\) and \((-2, 0)\)

Putting \( x = 0 \) to find where the curve crosses the y-axis

\( y = 2 \)

so the curve crosses the y-axis at \((0, 2)\)